III. THREE BASIC EXAMPLES IN MECHANICS

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A. Equations of motion

The equations of motion of a particle of mass m subject to force $\vec{F}$ are

$$d\vec{x} = \vec{v}\, dt \ ,$$

$$d\vec{v} = \frac{d\vec{p}}{m} = \frac{\vec{F}}{m}\, dt \ .$$

Or, dividing by dt

$$\frac{d\vec{x}}{dt} = \vec{v} \ , \qquad (1)$$

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} \ . \qquad (2)$$

Remember: dt always means a very small time interval, and for any quantity Q, dQ just means the change in Q during the time dt

$$dQ = Q(t + dt) - Q(t) \ .$$

dQ/dt is the change of Q per unit time.

The purpose of this lecture is to describe three basic examples of mechanics – projectile, harmonic oscillator, and planet.

B. Example 1: projectile motion

The first example is motion of a projectile, moving near the earth's surface, neglecting the effect of air resistance. (We'll study air resistance in Lecture 4.) See Figure 3.1.

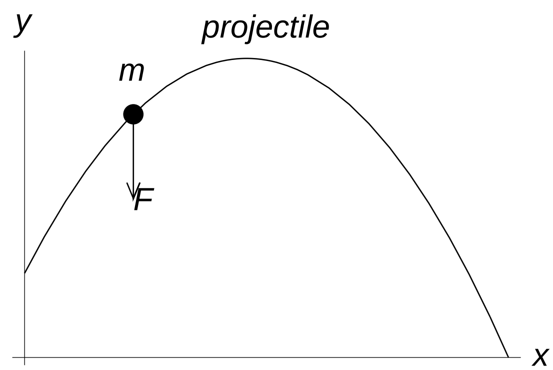

FIG. 1. Motion of a projectile. The only force on m is the gravitational force $\vec{F} = -mg\hat{\jmath}$.

The variables are

$$x, y = \text{ Cartesian coordinates,}$$

$$v_x, v_y = \text{ velocity components.}$$

The force components are

$$F_x = 0 \quad , \quad F_y = -mg \ ,$$

where m is the mass of the projectile, and g is the acceleration due to gravity

$$g = 9.8 \text{ m/s}^2 \ ;$$

the force is just the force of gravity, with magnitude mg and direction downward.

The equations of motion are

$$\frac{dx}{dt} = v_x \quad , \quad \frac{dv_x}{dt} = 0$$

$$\frac{dy}{dt} = v_y \quad , \quad \frac{dv_y}{dt} = -g \ .$$

We see from the equations that the horizontal motion (x and v_x) and vertical motion (y and v_y) are independent of each other. The analytic solution of the equations of motion is

$$x(t) = x_0 + v_{0x} t$$

$$v_x(t) = v_{0x}$$

$$y(t) = y_0 + v_{0y} t - \frac{1}{2} g\, t^2$$

$$v_y(t) = v_{0y} - gt$$

where $\vec{x}_0$ and $\vec{v}_0$ are the initial position and velocity. The horizontal component of motion has constant velocity, by Newton's $1st$ law, because the horizontal force component is zero. The vertical component of motion has constant acceleration $-g$, by Newton's $2nd$ law.

Exercise 1 For the projectile:

(a) Use a graphing calculator to plot $x(t)$, $y(t)$, and $y(x)$, for initial values

$$\vec{x}_0 = (0, 1) \text{ m} , \quad \vec{v}_0 = (34, 34) \text{ m/s} .$$

(b) Read from the graph the horizontal distance traveled when $y = 0$. Then calculate this distance by paper and pencil.

Comments

We may note these well-known properties of projectile motion; the graph of ...

- x *vs* t is a straight line with slope v_x.
- y *vs* t is a parabola.
- y *vs* x is a parabola.

Energy in projectile motion

The energy, which is a constant of the motion, is

$$E = \frac{1}{2} m (v_x^2 + v_y^2) + mgy .$$

For example, as m falls its potential energy (mgy) decreases, and its kinetic energy ($\frac{1}{2} m v_y^2$) increases, such that E is constant.

C. Example 2: harmonic oscillator

The second example is motion of a harmonic oscillator, *e.g.* a mass attached to a spring. See Figure 3.2.

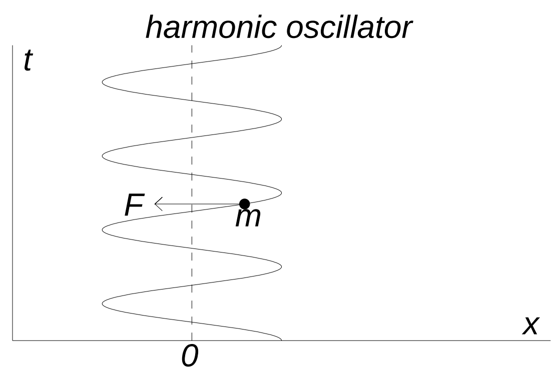

FIG. 2. Motion of a harmonic oscillator. The force on m is the restoring force $F = -kx$.

The variables are

$$x = \text{distance from equilibrium} ,$$

$$v = \text{velocity} .$$

The force is

$$F = -kx ,$$

where k is a constant. This force is a *restoring force*, always directed toward the equilibrium point $x = 0$. If $x > 0$, *i.e.* the spring is stretched, then $F < 0$, *i.e.* the mass is pulled back; if $x < 0$, *i.e.* the spring is compressed, then $F > 0$, *i.e.* the mass is pushed forward. The equation $F = -kx$, which is called *Hooke's law*, is a good approximation of the force exerted by a spring. The larger the displacement from $x = 0$, the stronger the restoring force.

The equations of motion are

$$\frac{dx}{dt} = v ,$$

$$\frac{dv}{dt} = -\frac{k}{m}x \ .$$

The analytic solution is

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$

$$v(t) = v_0 \cos(\omega t) - x_0 \omega \sin(\omega t)$$

where x_0 and v_0 are the initial position and velocity, and ω is

$$\omega = \sqrt{\frac{k}{m}} \ .$$

The position and velocity oscillate sinusoidally. The period of oscillation is

$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \ .$$

(ω is called the *angular frequency*.)

An interesting property of the harmonic oscillator is that the period τ does not depend on the amplitude of oscillation. We say the oscillation is *isochronous.* This property was discovered for pendulums by Galileo, reportedly by observing a swinging lamp in the cathedral at Pisa. (However, pendulums are only approximately isochronous, for small amplitude oscillations!)

Exercise 2 For the harmonic oscillator:

(a) Use a graphing calculator to plot $x(t)$ and $v(t)$ for initial values

$$x_0 = 1 \text{ cm} \ , \quad v_0 = 0 \ .$$

Let the spring constant k be 10.966 N/m and the mass m be 0.1 kg.

(b) Read from the graph the value of τ. Then calculate τ by paper and pencil.

Comments

We may note these well-known properties of the harmonic oscillator; the graph of ...

- x *vs* t is sinusoidal, *i.e.* harmonic in time.
- v *vs* t is sinusoidal, 90 degrees out of phase with x.

Energy in simple harmonic motion

The energy, which is a constant of the motion, is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \ .$$

As m moves away from the equilibrium point ($x = 0$) the spring either stretches or compresses, so the potential energy increases; the mass decelerates, because of the spring force, so the kinetic energy decreases; the total energy E remains constant.

D. Example 3: planetary orbit

The third example is motion of a planet in orbit around the sun. See Figure 3.3.

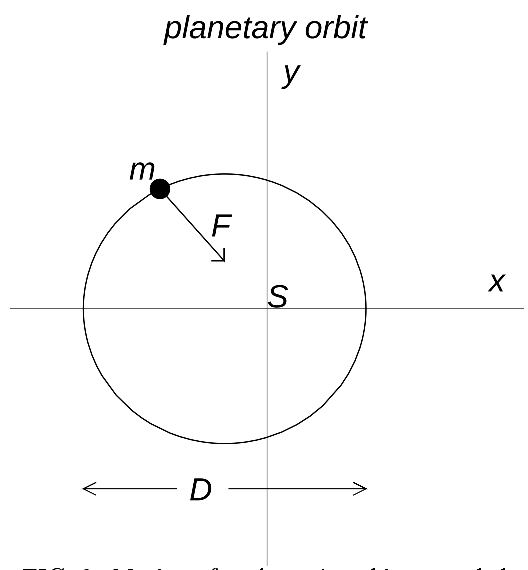

FIG. 3. Motion of a planet in orbit around the sun. The force is $\vec{F} = -GMm\hat{r}/r^2$. The orbit is an ellipse with parameters α and ϵ. Note that $x = r\cos\theta$ and $y = r\sin\theta$.

The gravitational force exerted by the sun on the planet is

$$\vec{F}(\vec{r}) = \frac{-GMm}{r^2}\,\hat{r} \qquad (3)$$

where $\vec{r}$ is the vector from the sun to the planet, $r = |\vec{r}|$ is the distance between the sun and the planet, and $\hat{r} =$

$\vec{r}/|\vec{r}|$ is the unit vector pointing from the sun toward the planet. Also, m is the mass of the planet, M is the mass of the sun, and G is Newton's gravitational constant. The value of GM is

$$GM = 1.33 \times 10^{20}\ \mathrm{m^3/s^2} = 39.5\ \mathrm{AU^3/yr^2}\ .$$

(1 astronomical unit (AU) is the mean distance from the Earth to the sun, a useful unit for planetary orbit calculations.) The orbit lies in a plane with coordinates x, y, and with velocity components v_x, v_y.

Aside Eq. (3) is Newton's Law of Universal Gravitation. When NASA rocket scientists plan a space flight, they use this equation to calculate the orbit of the satellite. So Newton, who came to understand gravity three hundred years ago, supposedly by contemplating a falling apple and the motion of the moon, is still relevant to the highest technological efforts of today.

The equations of motion are

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_x}{dt} = \frac{-GM\,x}{(x^2+y^2)^{3/2}}$$

$$\frac{dv_y}{dt} = \frac{-GM\,y}{(x^2+y^2)^{3/2}}$$

The analytic solution of these equations is that the orbit is an ellipse with the sun at one focus.

The equations for the orbit (taking the x-axis to be the major axis of the ellipse) are

$$x = r\cos\theta\ ,\quad y = r\sin\theta$$

where

$$r = \frac{\alpha}{1+\epsilon\cos\theta}\ , \qquad (4)$$

depending on 2 constants, α and ϵ. α is a measure of the radius of the orbit, and ϵ is the eccentricity of hte ellipse. The eccentricity in Fig 3.3 is 0.3.

Constants of the motion

There are *two* constants in planetary motion: the energy

$$E = \frac{1}{2}m(v_x^2+v_y^2) - \frac{GMm}{\sqrt{x^2+y^2}}\ , \qquad (5)$$

and the angular momentum

$$L = m(xv_y - yv_x)\ . \qquad (6)$$

We can relate the orbit parameters α and ϵ to the constants of the motion, by considering perihelion (p) and aphelion (a). At p or a, we have $v_x = 0$; thus

$$L = \frac{m\alpha|v_y(p)|}{1+\epsilon}\ ,\quad E = \frac{1}{2}mv_y^2(p) - \frac{GMm(1+\epsilon)}{\alpha}$$

$$L = \frac{m\alpha|v_y(a)|}{1-\epsilon}\ ,\quad E = \frac{1}{2}mv_y^2(a) - \frac{GMm(1-\epsilon)}{\alpha}$$

which gives us 4 equations for 4 unknowns (α,ϵ, $v_y(p)$,$v_y(a)$). After some algebra we find that

$$\alpha = \frac{L^2}{GMm^2} \qquad (7)$$

and

$$\epsilon = \sqrt{1+\frac{2\alpha E}{GMm}}\ . \qquad (8)$$

Exercise 3: Sending a satellite to Mars

The way to send a satellite to Mars, using the least amount of fuel, is to fire the rocket for a short time near the Earth, to put the satellite in an elliptical orbit around the sun with perihelion equal to Earth's orbit radius (1 AU) and aphelion equal to Mars's orbit radius (1.5 AU).

Use a graphing calculator to plot the orbit of a satellite for initial values

$$\vec{x}_0 = (1,0)\ \mathrm{AU}$$

$$\vec{v}_0 = (0, 6.88)\ \mathrm{AU/yr}$$

The easiest graphing technique is to use the *polar plot* feature, with $r(\theta)$ from Eq. (4). (HINT: First calculate

α and ϵ, by substituting the initial position and velocity into Eqs. (5) to (8).)

<u>Time dependence</u> Equation (4) describes the spatial orbit, but has no information about time. The following *parametric equations* describe the time dependence:

$$
\begin{aligned}
t &= \frac{\tau}{2\pi}(\psi - \epsilon \sin\psi) \ , \\
\theta &= 2 \arctan\left(\sqrt{\frac{1+\epsilon}{1-\epsilon}} \tan\frac{\psi}{2} \right) \ , \\
r &= \frac{\alpha(1-\epsilon\cos\psi)}{1-\epsilon^2} \ .
\end{aligned}
$$

The independent parameter ψ is called the eccentric anomaly. The equations for time dependence are complicated, but not beyond comprehension!

The period of revolution τ is

$$\tau = \frac{2\pi}{\sqrt{GM}} \left(\frac{\alpha}{1-\epsilon^2} \right)^{3/2} \ .$$

The length of the *major axis* of the ellipse is $D = 2\alpha/(1-\epsilon^2)$, so τ is simply related to D

$$\tau = \frac{2\pi}{\sqrt{GM}} \left(\frac{D}{2} \right)^{3/2} \ . \tag{9}$$

The relation $\tau \propto D^{3/2}$ is Kepler's third law of planetary motion, deduced by Kepler from empirical observations. We can use Eq. (9) to calculate how long it takes a satellite to travel from Earth to Mars: the major axis of the orbit is $D = 2.5$ AU; the travel time is a half period, 255 days.

<u>Comments</u>

We may note these well-known properties of the planetary orbit:

- The graph of y *vs* x is an ellipse, with the sun at one focus.
- The speed is maximum at perihelion and minimum at aphelion; by conservation of angular momentum

$$\frac{|v_y(p)|}{|v_y(a)|} = \frac{1+\epsilon}{1-\epsilon} \ .$$

E. A philosophical question

Why is it important to know how to calculate the orbit of a comet?

COMET

One possible answer to this question might be called the inventor's answer: In order to invent useful machines I need a true knowledge of mechanics, and the motion of a comet is a test of my equations of mechanics. But this answer is somehow unsatisfactory, because after all comets are never used in machines.

Another possible answer might be called the scientist's answer: I am curious about how the universe works, and I can learn something about that from the motion of a comet. But most people would consider this to be just *idle* curiosity.

Another possible answer might be called the philosopher's answer: Any knowledge is worthwhile, and ignorance breeds problems.

People used to fear comets, and believe that a comet portends disaster. The origin of the word "disaster" means *opposed by the stars.* A comet, a heavenly body that suddenly appears moving toward the sun and earth, might well be interpreted as a portent in the stars, to people who believe that heavenly bodies foretell the future. But, on the contrary, because we can *calculate* the orbit of a comet, the comet can't really portend very much: its motion is prescribed, controlled by objective mathematical laws.

Daniel Defoe wrote <u>A Journal of the Plague Year</u>, a fictional account of the real epidemic of bubonic plague that occured in London in 1665, an epidemic in which 75,000 people died. When Londoners became aware of the first cases of plague, and realized from past experience that a terrible epidemic was inevitable, they were beset by illogical beliefs. Defoe describes vividly how they were preyed upon by astrologers, quacks, and doomsayers. He wrote of a comet that had been observed shortly before the first infections, which many took as a portent, and of his own reaction to the comet:

`` ``In the first place, a blazing star or comet appeared for several months before the plague, as there did the year after another, ``