

# Introduction to Probability

*WARNING: Do not share this knowledge with any friends traveling to Las Vegas.*

The first lab in this course deals with probability. Specifically, it deals with probability distributions for 12 tossed dice; but the ideas can be generalized to many applications.

Suppose we toss a coin in the air and let it land. Clearly, there is a 50% probability that it will land heads up and a 50% probability that it will land heads down. By this we mean that if we continue tossing the coin repeatedly, the fraction of time that the coin lands heads up will approach  $\frac{1}{2}$ .

Suppose we toss two coins into the air at the same time. There are now 4 possible outcomes: both coins can have heads up, both heads down and two mixtures of heads and tails, depending on which coin is heads up. Since each of these outcomes is equally probable, the probability for each is  $\frac{1}{4}$  or 25%. Note that the total probability is  $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$ , or 100%. The sum of all probabilities has to add up to 1 because *something* has to happen. Note that the probability of both coins giving heads, or the probability of both coins giving tails is 25%, but the probability of getting one head and one tail is 50%. The factor of 2 larger probability is due to the fact that there are two possible ways of getting this combination.

# Binomial Distribution

These same basic ideas apply to rolling dice. Each die has 6 faces. So if I toss one die, the probability of any particular face being on top is  $1/6$ . In other words, there is a  $1/6$  probability of coming up 1, a  $1/6$  probability of coming up 2, etc. If we have 12 dice, as in this experiment, things are a bit more complicated – in the same way that having two coins is more complicated than having one coin.

When 12 dice are thrown, we can write the probability of obtaining  $n$  dice with a particular face up (say the red face) as:

$$B(n) = P^n(1-P)^{12-n} \frac{12!}{n!(12-n)!} .$$

This is known as the binomial distribution and is applicable not just for dice but for any similar problem in probability. Here  $P$  is the probability of getting the red face on the throw of a single die (so  $P = 1/6$ ).  $1-P$  is the probability of **not** getting the red face on the throw of a single die ( $1-P = 5/6$ ). Note again that the total probability of something happening is 100% or 1. So, the probability of getting a single red face among the 12 dice is  $(1/6)^1$  times  $(5/6)^{11}$  times a factor that gives the number of different ways that 12 dice can be tossed and one of them can have a red face up. We call this factor the combinatorial factor. In this case it is  $[12 \times 11 \times 10 \dots] / [(1)(11 \times 10 \times 9 \dots)] = 12$ . This makes sense if you think about it. Since there are 12 dice, there are 12 ways that one of them can come up with a red face. The resulting probability is 0.271 or 27.1%, in agreement with the listing in Table 2. If  $n$  is greater than 1, say 6, the combinatorial factor is much larger (924), since there are so many more ways of arranging 6 dice out of 12. If  $n$  increases all the way up to 12, then the combinatorial factor is just 1, since there is only 1 arrangement of 12 dice, all having the red face up.

# Measurements

Why do a measurement? To find the “true” value of some quantity.

In the real world, there is an uncertainty in any measurement.

A good measurement has random deviations from the true value. For example, if you measure the length of an object 10 times, each time you will likely get a slightly different answer. But these values should be close to and clustered around the true length.

By repeating the measurement several times, the randomness can be averaged away, and

→ results get closer to true value.



A bad measurement has systematic errors. This means some data are way off. An example would be measuring the length of the object with one edge aligned to the 1 cm line – instead of the zero of the ruler.

The values will NOT be clustered around the true value. Instead they will be systematically shifted by 1 cm. Repeating and averaging won't help in this case.

→ The cause of the problem must be identified and fixed.



# Measurements – An example

In some cases the true value for a measurement can be calculated exactly, like today's 1<sup>st</sup> experiment and the example below.

Usually, the true value is not known precisely, like today's 2<sup>nd</sup> experiment.

## Example:

Consider flipping a coin 4 times. On average, how many times will it be heads? (The true answer = 2.)

The probability machine shown here has exactly the same characteristics. For example, flipping 4 tails in a row is just as unlikely as the marble in the machine falling to the left 4 times in a row. If you repeat the measurement a large number of times, you will find that the histogram forms a bell-shaped curve, known as a Gaussian (or normal) curve. This curve is easily derived from the more general binomial distribution formula.

