

PHYS851 Quantum Mechanics I, Fall 2009  
 HOMEWORK ASSIGNMENT 10: Solutions

**Topics Covered:** Tensor product spaces, change of coordinate system, general theory of angular momentum

**Some Key Concepts:** Angular momentum: commutation relations, raising and lowering operators, eigenstates and eigenvalues.

1. [10 pts] Consider the position eigenstate  $|\vec{r}\rangle$ . In spherical coordinates, this state is written as  $|r\theta\phi\rangle$ , where  $\vec{R}|r\theta\phi\rangle = r\vec{e}_r(\theta, \phi)|r\theta\phi\rangle$ . In cartesian coordinates, the same state is written  $|xyz\rangle$ , where  $\vec{R}|xyz\rangle = (x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)|xyz\rangle$ . Evaluate the following:

Most of these can be evaluated in many different ways, I am just giving one possibility for each:

$$(a) \langle r\theta\phi|r'\theta'\phi'\rangle = \frac{1}{r^2 \sin\theta} \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

$$(b) \langle r\theta\phi|xyz\rangle = \delta(r \sin\theta \cos\phi - x)\delta(r \sin\theta \sin\phi - y)\delta(r \cos\theta - z)$$

$$(c) \langle r\theta\phi|p_x p_y p_z\rangle = [2\pi\hbar]^{-3/2} \exp\left[\frac{i}{\hbar}(p_x r \sin\theta \cos\phi + p_y r \sin\theta \sin\phi + p_z r \cos\theta)\right]$$

$$(d) \langle r\theta\phi|\vec{R}|r'\theta'\phi'\rangle = \frac{1}{r \sin\theta} \vec{e}_r(\theta, \phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

$$(e) \langle r\theta\phi|Z|r'\theta'\phi'\rangle = \frac{\cot\theta}{r} \vec{e}_r(\theta, \phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')$$

$$(f) \langle r\theta\phi|P_z|r'\theta'\phi'\rangle = -i\hbar\partial_z \langle r\theta\phi|r'\theta'\phi'\rangle$$

$$\text{now } \partial_z = \frac{\partial r}{\partial z} \partial_r + \frac{\partial \theta}{\partial z} \partial_\theta + \frac{\partial \phi}{\partial z} \partial_\phi = \sec\theta \partial_r - \frac{1}{r \sin\theta} \partial_\theta$$

so that

$$\begin{aligned} \langle r\theta\phi|P_z|r'\theta'\phi'\rangle &= 2i\hbar \frac{\sec\theta \csc\theta}{r^3} \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') - i\hbar \frac{\sec\theta \csc\theta}{r^2} \delta'(r-r')\delta(\theta-\theta')\delta(\phi-\phi') \\ &\quad - i\hbar \frac{\cos\theta \csc^3\theta}{r^3} \delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi') + i\hbar \frac{\csc^2\theta}{r^3} \delta(r-r')\delta'(\theta-\theta')\delta(\phi-\phi') \end{aligned}$$

2. [10 pts] Consider a system consisting of two spin-less particles with masses  $m_1$  and  $m_2$ , and charges  $q_1$  and  $q_2$ .

(a) Write the quantum mechanical Hamiltonian that describes this system.

$$H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{R}_1 - \vec{R}_2|}$$

(b) Define a suitable tensor-product-state basis to describe the system.

Basis:  $\{|\vec{r}_1, \vec{r}_2\rangle^{(12)}\}$  where  $|\vec{r}_1, \vec{r}_2\rangle^{(12)} = |\vec{r}_1\rangle^{(1)} \otimes |\vec{r}_2\rangle^{(2)}$

(c) Evaluate the expression  $\langle b|H|\psi\rangle$ , where  $|\psi\rangle$  is an arbitrary state of the system, and  $|b\rangle$  should be replaced by one of your basis states.

$$\langle \vec{r}_1, \vec{r}_2 | H | \psi \rangle = \left[ -\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + \frac{q_1 q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|} \right] \psi(\vec{r}_1, \vec{r}_2)$$

(d) Do the same for  $N$  identical particles of mass  $m$  and charge  $q$ .

Basis:  $\{|\vec{r}_1, \dots, \vec{r}_N\rangle\}$ , where  $|\vec{r}_1, \dots, \vec{r}_N\rangle = |\vec{r}_1\rangle^{(1)} \otimes \dots \otimes |\vec{r}_N\rangle^{(N)}$

$$\langle \vec{r}_1, \dots, \vec{r}_N | H | \psi \rangle = \sum_{j=1}^N \left[ -\frac{\hbar^2}{2m_j} \nabla_j^2 + \sum_{k=j+1}^N \frac{q_j q_k}{4\pi\epsilon_0 |\vec{r}_j - \vec{r}_k|} \right] \psi(\vec{r}_1, \dots, \vec{r}_N)$$

3. [20 pts] Let  $\vec{L} = \vec{R} \times \vec{P}$ , where  $\vec{L}$ ,  $\vec{R}$ , and  $\vec{P}$  are the three-dimensional vector operators for angular momentum, position, and linear momentum, respectively. For  $\mu, \nu \in \{x, y, z\}$ , evaluate the following expressions:

(a)  $[R_\mu, R_\nu] = 0$

(b)  $[P_\mu, P_\nu] = 0$

(c)  $[R_\mu, P_\nu] = i\hbar\delta_{\mu,\nu}$

Use these results to prove explicitly that  $[L_x, L_y] = i\hbar L_z$ , then use a symmetry argument to obtain similar expressions for the commutators  $[L_y, L_z]$  and  $[L_x, L_z]$ .

We start from  $L_x = YP_z - ZP_y$ ,  $L_y = ZP_x - XP_z$ , and  $L_z = XP_y - YP_x$ , so that

$$\begin{aligned}
 [L_x, L_y] &= [YP_z - ZP_y, ZP_x - XP_z] \\
 &= [YP_z, ZP_x] - [YP_z, XP_z] - [ZP_y, ZP_x] + [ZP_y, XP_z] \\
 &= YP_x[P_z, Z] - 0 - 0 + P_yX[Z, P_z] \\
 &= -i\hbar YP_x + i\hbar XP_y \\
 &= i\hbar L_z
 \end{aligned}$$

For any right-handed triplet of unit-vectors, the choice which to call  $\vec{e}_x$ , is arbitrary, but once the choice is made, the labels of the other two are fixed due to handedness. Thus cyclic permutations of  $x, y, z$  are all equivalent to different labeling choices. This means that with different choices, our result would become  $[L_y, L_z] = i\hbar L_x$  or  $[L_z, L_x] = i\hbar L_y$ .

4. [10 pts] Show explicitly that  $J^2 = J_x^2 + J_y^2 + J_z^2$  commutes with  $J_z$ , then use a symmetry argument to show that  $J^2$  must also commute with  $J_x$  and  $J_y$ . Then, answer the following (be sure to explain your reasoning):

$$\begin{aligned}
 [J_x^2, J_z] &= J_x^2 J_z - J_z J_x^2 \\
 &= J_x^2 J_z - J_x J_z J_x + J_x J_z J_x - J_z J_x^2 \\
 &= J_x [J_x, J_z] + [J_x, J_z] J_x \\
 &= -i\hbar J_x J_y - i\hbar J_y J_x
 \end{aligned}$$

$$\begin{aligned}
 [J_y^2, J_z] &= J_y^2 J_z - J_z J_y^2 \\
 &= J_y^2 J_z - J_y J_z J_y + J_y J_z J_y - J_z J_y^2 \\
 &= J_y [J_y, J_z] + [J_y, J_z] J_y \\
 &= i\hbar J_y J_x - i\hbar J_x J_y
 \end{aligned}$$

$$[J_z^2, J_z] = 0$$

Adding these results together gives

$$[J^2, J_z] = 0$$

- (a) Do simultaneous eigenstates of  $J_x$  and  $J_z$  exist?

No, because  $J_x$  and  $J_z$  do not commute.

- (b) Do simultaneous eigenstates of  $J^2$  and  $J_z$  exist?

Yes, because  $J^2$  and  $J_z$  commute.

- (c) Do simultaneous eigenstates of  $J^2$  and  $J_y$  exist?

Yes, because if  $J^2$  commutes with  $J_z$ , then by symmetry, it must commute with  $J_y$ .

- (d) Do simultaneous eigenstates of  $J^2$ ,  $J_z$ , and  $J_y$  exist?

No, because  $J_z$  and  $J_y$  do not commute.

- (e) Do simultaneous eigenstates of  $J^2$  and  $J_x^2$  exist?

Yes, simultaneous eigenstates of  $J^2$  and  $J_x$  exist because  $J^2$  and  $J_x$  commute. Clearly an eigenstate of  $J_x$  is also an eigenstate of  $J_x^2$ .

- (f) Do simultaneous eigenstates of  $J_z$  and  $J_x^2 + J_y^2$  exist?

Yes, because  $J_x^2 + J_y^2 = J^2 - J_z^2$ , and  $J_z$  commutes with  $J^2$  and  $J_z^2$ .

5. [20 pts] With  $J_{\pm} = J_x \pm iJ_y$ , express  $J_+J_-$  and  $J_-J_+$  in terms of the operators  $J^2$  and  $J_z$ , then compute the the following commutators:

$$\begin{aligned} J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 + J_y^2 + i[J_y, J_x] \\ &= J^2 - J_z^2 + \hbar J_z \end{aligned}$$

likewise

$$J_-J_+ = J^2 - J_z^2 - \hbar J_z$$

(a)  $[J_+, J_-]$

$$[J_+, J_-] = J_+J_- - J_-J_+ = 2\hbar J_z$$

(b)  $[J_{\pm}, J^2]$

$$[J_{\pm}, J^2] = [J_x, J^2] \pm i[J_y, J^2] = 0$$

(c)  $[J_{\pm}, J_z]$

$$[J_{\pm}, J_z] = [J_x, J_z] \pm i[J_y, J_z] = -i\hbar J_y \pm (-\hbar J_x) = -\hbar J_{\pm}$$

(d)  $[J_{\pm}, J_x]$

$$[J_{\pm}, J_x] = \pm i[J_y, J_x] = \pm \hbar J_z$$

(e)  $[J_{\pm}, J_y]$

$$[J_{\pm}, J_y] = [J_x, J_y] = i\hbar J_z$$



7. [10 pts] Consider a system described by the Hamiltonian

$$H = \Delta J_z + U(J_x^2 + J_y^2) \quad (1)$$

where  $J_x$ ,  $J_y$ , and  $J_z$  are the three components of a generalized angular momentum operator, and  $\Delta$  and  $U$  are constants. What are the energy levels of this system?

With  $J_x^2 + J_y^2 = J^2 - J_z^2$ , the Hamiltonian becomes

$$H = U(J^2 - J_z^2) + \Delta J_z$$

The eigenstates of this are the standard  $|j, m\rangle$  states, with eigenvalues given by

$$H|j, m\rangle = (\hbar^2 U(j^2 + j - m^2) + \hbar \Delta m) |j, m\rangle$$