PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 11

Topics Covered: Orbital angular momentum, center-of-mass coordinates
Some Key Concepts: angular degrees of freedom, spherical harmonics

1. [20 pts] In order to derive the properties of the spherical harmonics, we need to determine the action of the angular momentum operator in spherical coordinates. Just as we have $\langle x|P_x|\psi\rangle = -i\hbar\frac{d}{dx}\langle x|\psi\rangle$, we should find a similar expression for $\langle r\theta\phi|\vec{L}|\psi\rangle$. From $\vec{L} = \vec{R}\times\vec{P}$ and our knowledge of momentum operators, it follows that

$$\langle r\theta\phi|\vec{L}|\psi\rangle = -\imath\hbar\left(\vec{e}_x\left(y\frac{d}{dz} - z\frac{d}{dy}\right) + \vec{e}_y\left(z\frac{d}{dx} - x\frac{d}{dz}\right) + \vec{e}_z\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\right)\langle r\theta\phi|\psi\rangle.$$

Cartesian coordinates are related to spherical coordinates via the transformations

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

and the inverse transformations

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \arctan(\frac{\sqrt{x^2+y^2}}{z})$$
$$\phi = \arctan(\frac{y}{x}).$$

Their derivatives can be related via expansions such as

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\phi}{\partial x}\partial_\phi.$$

Using these relations, and similar expressions for ∂_y and ∂_z, find expressions for $\langle r\theta\phi|L_x|\psi\rangle$, $\langle r\theta\phi|L_y|\psi\rangle$, and $\langle r\theta\phi|L_z|\psi\rangle$, involving only spherical coordinates and their derivatives.

2. [15pts] From your previous answer and the definition $L^2 = L_x^2 + L_y^2 + L_z^2$, prove that

$$\langle r\theta\phi|L^2|\psi\rangle = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi^2}\right)\langle r\theta\phi|\psi\rangle.$$

3. [10 pts] We can factorize the Hilbert space of a 3-D particle into radial and angular Hilbert spaces, $\mathcal{H}^{(3)} = \mathcal{H}^{(r)}\otimes\mathcal{H}^{(\Omega)}$. Two alternate basis sets that both span $\mathcal{H}^{(\Omega)}$ are $\{|\theta\phi\rangle\}$ and $\{|\ell m\rangle\}$. As the angular momentum operator lives entirely in $\mathcal{H}^{(\Omega)}$, we can use our results from problem 11.1 to derive an expression for $\langle\theta\phi|L_z|\ell m\rangle$. Combine this with the formula $L_z|\ell m\rangle = \hbar m|\ell m\rangle$, to derive and then solve a differential equation for the ϕ-dependence of $\langle\theta\phi|\ell m\rangle$. Your solution should give $\langle\theta\phi|\ell m\rangle$ in terms of the as of yet unspecified initial condition $\langle\theta|\ell m\rangle \equiv \langle\theta,\phi|\ell m\rangle\Big|_{\phi=0}$. What restrictions does this solution impose on the quantum number m, which describes the z-component of the orbital angular momentum? Since $m_{max} = \ell$, what restrictions are then placed on the total angular momentum quantum number ℓ?

4. [10 pts] Using $L_\pm = L_x \pm iL_y$ we can use the relation $L_+|\ell,\ell\rangle = 0$ and the expressions from problem 11.1 to write a differential equation for $\langle\theta\phi|\ell\ell\rangle$. Plug in your solution from 11.3 for the ϕ-dependence, and show that the solution is $\langle\theta\phi|\ell\ell\rangle = c_\ell e^{i\ell\phi}\sin^\ell(\theta)$. Determine the value of the normalization coefficient c_ℓ by performing the necessary integral.

5. [10 pts] Using $L_-|\ell m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}|\ell, m-1\rangle$ together with your previous answers to derive an expression for $\langle\theta\phi|\ell, m-1\rangle$ in terms of $\langle\theta\phi|\ell m\rangle$. Explain how in principle you can now recursively calculate the value of the spherical harmonic $Y_\ell^m(\theta\phi) \equiv \langle\theta\phi|\ell m\rangle$ for any θ and ϕ and for any ℓ and m. Follow your procedure to derive properly normalized expressions for spherical harmonics for the case $\ell = 1$, $m = -1, 0, 1$.

6. [10 pts] A particle of mass M is constrained to move on a spherical surface of radius a.

   Does the system live in $\mathcal{H}^{(3)}$, $\mathcal{H}^{(r)}$, or $\mathcal{H}^{(\Omega)}$? What is the Hamiltonian? What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

7. [10 pts] Two particles of mass M_1 and M_2 are attached to a massless rigid rod of length d. The rod is attached to an axle at its center-of-mass, and is free to rotate without friction in the x-y plane.

   Describe the Hilbert space of the system and then write the Hamiltonian. What are the energy levels and degeneracies? What are the wavefunctions of the energy eigenstates?

8. [10 pts] For a two-particle system, the transformation to relative and center-of-mass coordinates is defined by

$$\vec{R} = \vec{R}_1 - \vec{R}_2$$

$$\vec{R}_{CM} = \frac{m_1\vec{R}_1 + m_2\vec{R}_2}{m_1 + m_2}$$

   The corresponding momenta are defined by

$$\vec{P} = \mu\frac{d}{dt}\vec{R}$$

$$\vec{P}_{CM} = M\frac{d}{dt}\vec{R}_{CM}$$

   where $\mu = m_1 m_2/M$ is the reduced mass, and $M = m_1 + m_2$ is the total mass. Invert these expressions to write $\vec{R}_1$, $\vec{R}_2$, $\vec{P}_1$, and $\vec{P}_2$ in terms of $\vec{R}$, $\vec{R}_{CM}$, $\vec{P}$, and $\vec{P}_{CM}$.