PHYS851 Quantum Mechanics I, Fall 2009
HOMEWORK ASSIGNMENT 13: Solutions

1. In this problem you will derive the 2×2 matrix representations of the three spin observables from first principles:

   (a) In the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$, the matrix representation of S_z is of course

$$S_z = \begin{pmatrix} \langle\uparrow_z|S_z|\uparrow_z\rangle & \langle\uparrow_z|S_z|\downarrow_z\rangle \\ \langle\downarrow_z|S_z|\uparrow_z\rangle & \langle\downarrow_z|S_z|\downarrow_z\rangle \end{pmatrix}. \tag{1}$$

   Use Eqs. (1) and (2) to find the four matrix elements of S_z in the basis of its own eigenstates.

   From $S_z|\uparrow_z\rangle = \frac{\hbar}{2}|\uparrow_z\rangle$ and the orthonormality of the basis, it follows that $\langle\uparrow_z|S_z|\uparrow_z\rangle = \frac{\hbar}{2}$ and $\langle\downarrow_z|S_z|\uparrow_z\rangle = 0$.
   From $S_z|\downarrow_z\rangle = -\frac{\hbar}{2}|\downarrow_z\rangle$ and the orthonormality of the basis, it follows that $\langle\uparrow_z|S_z|\downarrow_z\rangle = 0$ and $\langle\downarrow_z|S_z|\downarrow_z\rangle = -\frac{\hbar}{2}$.
   This gives us

$$S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

   (b) Invert the definitions $S_+ = S_x + iS_y$ and $S_- = S_x - iS_y$, to express S_x and S_y in terms of S_+ and S_-.

   Inverting these equations gives

$$S_x = \frac{1}{2}(S_+ + S_z) \tag{3}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \tag{4}$$

   (c) Use the equation

$$S_\pm|s, m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s\pm1)}|s, m_s\pm1\rangle \tag{5}$$

   to find the matrix elements of S_+ and S_- in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle$.

   This formula gives $S_+|\uparrow_z\rangle = 0$, $S_+|\downarrow_z\rangle = \hbar|\uparrow_z\rangle$, so that orthonormality gives

$$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6}$$

   Likewise, $S_-|\uparrow_z\rangle = \hbar|\downarrow_z\rangle$ and $S_-|\downarrow_z\rangle = 0$, so that

$$S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{7}$$

(d) From your answers to 13.1.b and 13.1.c, derive the matrix representations of S_x and S_y for spin-1/2.

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{9}$$

(e) Explicitly verify that these operators satisfy the angular momentum commutation relations.

$$\begin{aligned} [S_x, S_y] &= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\ &= \frac{\hbar^2}{4}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) \\ &= i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar S_z \end{aligned} \tag{10}$$

$$\begin{aligned} [S_y, S_z] &= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) \\ &= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right) \\ &= i\frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar S_x \end{aligned} \tag{11}$$

$$\begin{aligned} [S_z, S_x] &= \frac{\hbar^2}{4}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ &= i\frac{\hbar^2}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= i\hbar S_y \end{aligned} \tag{12}$$

(f) Show explicitly that $S^2 = \hbar^2 s(s+1)I$.

$$\begin{aligned} S^2 &= S_x^2 + S_y^2 + S_z^2 && (13)\\ &= \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \\ &= \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \hbar^2 s(s+1)I && (14) \end{aligned}$$

(g) Based on symmetry, write the 2×2 matrix representations of S_x, S_y, and S_z in the basis of eigenstates of S_y.

A cyclic permutation (relabeling) of the indices leaves the commutation relations unchanged. Thus we can relabel the indices according to $x' = y$, $y' = z$m and $z' = x$. Thus the eigenstates of y' are just the eigenstates of z, in terms of the primed indices we have

$$\begin{aligned} S_{x'} &= S_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && (15)\\ S_{y'} &= S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (16)\\ S_{z'} &= S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && (17) \end{aligned}$$

Note that the eigenstates of S_y are only defined up to a phase-factor, different phase factor choices will lead to different sets of matrices. The above result, which reflects the permutation symmetry of the angular momentum group, will be generated by setting the phase factors so that

$$\begin{aligned} |\uparrow_y\rangle &= \frac{e^{i\phi}}{\sqrt{2}}(|\uparrow_z\rangle + i|\downarrow_z\rangle) && (18)\\ |\downarrow_y\rangle &= \frac{e^{i\phi}}{\sqrt{2}}(|\uparrow_z\rangle - i|\downarrow_z\rangle) && (19) \end{aligned}$$

where ϕ is an arbitrary phase.

2. **Pauli spin matrices:** The Pauli spin matrices, σ_x, σ_y, and σ_z are defined via

$$\vec{S} = \hbar s \vec{\sigma} \tag{20}$$

(a) Use this definition and your answers to problem 13.1 to derive the 2×2 matrix representations of the three Pauli matrices in the basis of eigenstates of S_z.

With $s = 1/2$, this gives

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{21}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{22}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

(b) For each Pauli matrix, find its eigenvalues, and the components of its normalized eigenvectors in the basis of eigenstates of S_z.

Each Pauli matrix has eigenvalues 1 and -1.
The eigenvectors are

$$|\uparrow_z\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

$$|\downarrow_z\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

$$|\uparrow_x\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{26}$$

$$|\downarrow_x\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{27}$$

$$|\uparrow_y\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{28}$$

$$|\downarrow_y\rangle \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{29}$$

(c) Use your answer to 13.2.b to obtain the eigenvalues of S_x, S_y, and S_z, as well as the components of the corresponding normalized eigenvectors in the basis of eigenstates of S_z.

Each component of $\vec{S}$ has eigenvalues $\hbar/2$ and $-\hbar/2$.
The eigenvectors are the same as in 13.2(b).

3. Repeat problems 13.1.(a-d) and 13.2.a for the case of a spin-1 particle.

For $s = 1$, the eigenvalues of S_z are 1, 0, and -1. Thus we can introduce the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$, defined by $S_z|m\rangle = \hbar m|m\rangle$.
In this basis we must have

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{30}$$

We still have $S_x = \frac{1}{2}(S_+ + S_-)$ and $S_y = \frac{1}{2i}(S_- - S_+)$.

From $S_\pm|m\rangle = \sqrt{s(s+1) - m(m\pm1)}|m\pm1\rangle$ we find

$$S_+|m\rangle = \hbar\sqrt{2-m(m+1)}|m+1\rangle \tag{31}$$
$$S_-|m\rangle = \hbar\sqrt{2-m(m-1)}|m-1\rangle \tag{32}$$

this leads to

$$S_+ = \begin{pmatrix} \langle 1|S_+|1\rangle & \langle 1|S_+|0\rangle & \langle 1|S_+|-1\rangle \\ \langle 0|S_+|1\rangle & \langle 0|S_+|0\rangle & \langle 0|S_+|-1\rangle \\ \langle -1|S_+|1\rangle & \langle -1|S_+|0\rangle & \langle -1|S_+|-1\rangle \end{pmatrix} = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

and

$$S_- = \begin{pmatrix} \langle 1|S_-|1\rangle & \langle 1|S_-|0\rangle & \langle 1|S_-|-1\rangle \\ \langle 0|S_-|1\rangle & \langle 0|S_-|0\rangle & \langle 0|S_-|-1\rangle \\ \langle -1|S_-|1\rangle & \langle -1|S_-|0\rangle & \langle -1|S_-|-1\rangle \end{pmatrix} = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{34}$$

which gives

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{35}$$

and

$$S_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{36}$$

The generalized Pauli matrices for spin-1 would then be

$$\sigma_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{37}$$

$$\sigma_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{38}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{39}$$

4. Consider an electron whose position is held fixed, so that it can be described by a simple two-component spinor (i.e. no $\vec{r}$ dependence). Let the initial state of the electron be spin up relative to the z-axis, $|\uparrow_z\rangle$. At time $t = 0$, a uniform magnetic field is applied along the y-axis. What is the state-vector of the electron at time $t > 0$?

Hint: Start by writing the Hamiltonian, which should contain only the spin-contribution to the magnetic dipole energy. Then propagate the state using the energy eigenvalue representation of the propagator, $U(t) = \sum_n |\omega_n\rangle e^{-i\omega_n t}\langle\omega_n|$.

The initial state of the system is $|\psi(0)\rangle = |\uparrow_z\rangle$
The Hamiltonian is

$$H = -\vec{\mu}\cdot\vec{B} = -\frac{gq}{2M}\vec{S}\cdot\vec{B} = \frac{|e|\hbar B_0}{2m_e}\sigma_y = \mu_B B_0 \sigma_y \tag{40}$$

With $\omega_0 = \frac{|e|B_0}{m_e}$, the eigenvalues of H are then $\pm\hbar\omega_0$.
The state at time t is then

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt/\hbar}|\psi(0)\rangle & (41)\\ &= \left(\cos(Ht/\hbar) - i\sin(Ht/\hbar)\right)|\psi_0\rangle & (42)\\ &= \left(\cos(\omega_0 t)I - i\sin(\omega_0 t)\sigma_y\right)|\uparrow_z\rangle & (43)\\ &= \cos(\omega_0 t)|\uparrow_z\rangle + \sin(\omega_0 t)|\downarrow_z\rangle & (44) \end{aligned}$$

5. Consider the most general normalized spin-1/2 state $|\psi\rangle = c_\uparrow|\uparrow_z\rangle + c_\downarrow|\downarrow_z\rangle$.

(a) Compute $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$, with respect to this state.

$$\langle S_x\rangle = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} c_- \\ c_+ \end{pmatrix} = \frac{\hbar}{2}(c_+^* c_- + c_-^* c_+)$$
$$\langle S_y\rangle = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} -ic_- \\ ic_+ \end{pmatrix} = \frac{\hbar}{2i}(c_+^* c_- - c_-^* c_+)$$
$$\langle S_z\rangle = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \frac{\hbar}{2}\left(c_+^*, c_-^*\right)\begin{pmatrix} c_+ \\ -c_- \end{pmatrix} = \frac{\hbar}{2}(c_+^* c_+ - c_-^* c_-)$$

(b) Compute the variances ΔS_x, ΔS_y, and ΔS_z.

We know that

$$\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle = \frac{\hbar^2}{4}.$$

So that

$$\Delta S_x = \sqrt{\langle S_x^2\rangle - \langle S_x\rangle^2} = \frac{\hbar}{2}\sqrt{1 - (c_+^* c_- + c_-^* c_+)^2}$$
$$\Delta S_y = \sqrt{\langle S_y^2\rangle - \langle S_y\rangle^2} = \frac{\hbar}{2}\sqrt{1 + (c_+^* c_- - c_-^* c_+)^2}$$
$$\Delta S_z = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \frac{\hbar}{2}\sqrt{1 - (c_+^* c_+ - c_-^* c_-)^2}$$

(c) Prove that $\Delta S_x = \frac{\hbar}{2}\left|c_\uparrow^2 - c_\downarrow^2\right|$, $\Delta S_y = \frac{\hbar}{2}\left|c_\uparrow^2 + c_\downarrow^2\right|$, and $\Delta S_z = \hbar|c_\uparrow||c_\downarrow|$.

With

$$(c_+^* c_+ + c_-^* c_-)^2 = 1^2 = 1$$

we can write ΔS_x as

$$\begin{aligned}
\Delta S_x &= \frac{\hbar}{2}\sqrt{(c_+^* c_+ + c_-^* c_-)^2 - (c_+^* c_- + c_-^* c_+)^2} \\
&= \frac{\hbar}{2}\sqrt{|c_+|^4 + 2|c_+|^2|c_-|^2 + |c_-|^4 - (c_+^*)^2 c_-^2 - 2|c_+|^2|c_-|^2 - (c_-^*)^2 c_+^2} \\
&= \frac{\hbar}{2}\sqrt{(c_+^*)^2 c_+^2 - (c_+^*)^2 c_-^2 - (c_-^*)^2 c_+^2 + (c_-^*)^2 c_-^2} \\
&= \frac{\hbar}{2}\sqrt{((c_+^*)^2 - (c_-^*)^2)(c_+^2 - c_-^2)} \\
&= \frac{\hbar}{2}|c_+^2 - c_-^2|
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\Delta S_y &= \frac{\hbar}{2}\sqrt{(c_+^*)^2 c_+^2 + 2c_+^* c_-^* c_+ c_- + (c_-^*)^2 c_-^2 + (c_+^*)^2 c_-^2 - 2c_+^* c_-^* c_+ c_- + (c_-^*)^2 c_+^2} \\
&= \frac{\hbar}{2}\sqrt{(c_+^*)^2 c_+^2 + (c_-^*)^2 c_-^2 + (c_+^*)^2 c_-^2 + (c_-^*)^2 c_+^2} \\
&= \frac{\hbar}{2}|c_+^2 + c_-^2|
\end{aligned}$$

and

$$\begin{aligned}\Delta S_z &= \frac{\hbar}{2}\sqrt{(|c_+|^2 + |c_-|^2)^2 - (|c_+|^2 - |c_-|^2)^2} \\ &= \frac{\hbar}{2}\sqrt{4|c_+|^2|c_-|^2} \\ &= \hbar|c_+||c_-|\end{aligned}$$

6. **The Stern-Gerlach effect:** A Stern-Gerlach analyzer (SGA) spatially separates the m_s states, relative to the axis of alignment, of any particle with spin sent through the device.

   (a) Consider an SGA aligned along the z-axis. At the location of the beam center, the magnetic field inside the SGA can be written to good approximation as $\vec{B}(\vec{r}) = B_0(z)\vec{e}_z$, where $B_0(z)$ is a monotonically increasing function of z. The operator for the spin magnetic dipole energy is $V_B = -\vec{\mu} \cdot \vec{B}(\vec{r})$. Due to conservation of energy, the electron must experience a force in the direction of decreasing dipole energy. Show that this force is orthogonal to the Lorentz force if the particle has charge, and that it will deflect the spin up and spin down states in opposite directions.

   The Lorentz force is $q\vec{v} \times \vec{B}$, and so is always perpendicular to $\vec{B}$. This means that any observed deflection along the z-axis can only be attributed to spin effects.
   The potential for spin-up relative to $\vec{B}$, is

$$V_\uparrow = \langle \uparrow | V_B | \uparrow \rangle = \mu_B B_0(z) \tag{45}$$

   and for spin-down

$$V_\downarrow = \langle \downarrow | V_B | \downarrow \rangle = -\mu_B B_0(z) \tag{46}$$

   This shows that the magnetic dipole energy of the spin-up state increases with increasing $B_0(z)$, and thus the spin-up state will experience a downward force.
   The magnetic dipole energy of the spin-down state decreases with increasing $B_0(z)$, and thus the spin-down state will experience an upward force.

   (b) A single electron in the $|\uparrow_z\rangle$ state, is directed into SGA1, which is aligned along the x-axis. Determine the probabilities for the electron to exit SGA1 in the $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ channels.

   The probabilities are

$$P_1(\uparrow) = |\langle \uparrow_z | \uparrow_x \rangle|^2 = \frac{1}{2} \tag{47}$$

$$P_1(\downarrow) = |\langle \uparrow_z | \downarrow_x \rangle|^2 = \frac{1}{2} \tag{48}$$

   (c) The output beam from SGA1 corresponding to the $|\downarrow_x\rangle$ channel is then directed into SGA2, which is aligned along the z-axis. While the output beam from SGA1 corresponding to the $|\uparrow_x\rangle$ channel is directed into SGA3, which is aligned along the unit vector $\frac{1}{\sqrt{2}}\vec{e}_z + \frac{1}{\sqrt{2}}\vec{e}_y$. Determine the probabilities for the electron to exit in each of the four output channels (i.e. two for SGA2 and two for SGA3).

   The independent probabilities for SGA2 are

$$P_2(\uparrow) = |\langle \uparrow_z | \downarrow_x \rangle|^2 = \frac{1}{2} \tag{49}$$

$$P_2(\downarrow) = |\langle \downarrow_z | \downarrow_z \rangle|^2 = \frac{1}{2} \tag{50}$$

   and for SGA3 we need to find the eigenvectors of

$$S_3 = \frac{1}{\sqrt{2}}(S_z + S_y) = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{51}$$

which must have eigenvalues $\pm\hbar/2$. The eigenvectors are solutions to

$$\frac{1}{\sqrt{2}}a - \frac{i}{\sqrt{2}}b = \pm a \tag{52}$$

with $a = 1$, we find $b = -i(1 \mp \sqrt{2})$, which after normalization gives

$$\begin{aligned} |\uparrow_3\rangle &= \frac{|\uparrow_z\rangle - i(1-\sqrt{2})|\downarrow_z\rangle}{\sqrt{1+(1-\sqrt{2})^2}} && (53) \\ |\downarrow_3\rangle &= \frac{|\uparrow_z\rangle - i(1+\sqrt{2})|\downarrow_z\rangle}{\sqrt{1+(1+\sqrt{2})^2}} && (54) \end{aligned}$$

The probabilities are then

$$\begin{aligned} P_3(\uparrow) &= |\langle\uparrow_x|\uparrow_3\rangle|^2 = \frac{1+(1-\sqrt{2})}{1+(1-\sqrt{2})^2} = \frac{2-\sqrt{2}}{4-2\sqrt{2}} = \frac{1}{2} && (55) \\ P_3(\downarrow) &= |\langle\uparrow_x|\downarrow_3\rangle|^2 = \frac{1+(1+\sqrt{2})}{1+(1+\sqrt{2})^2} = \frac{2+\sqrt{2}}{4+2\sqrt{2}} = \frac{1}{2} && (56) \end{aligned}$$

Computing the combined probabilities then gives

$$\begin{aligned} P(\uparrow_x) &= P_1(\downarrow)P_2(\uparrow) = \frac{1}{4} && (57) \\ P(\downarrow_x) &= P_1(\downarrow)P_2(\downarrow) = \frac{1}{4} && (58) \\ P(\uparrow_3) &= P_1(\uparrow)P_3(\uparrow) = \frac{1}{4} && (59) \\ P(\downarrow_3) &= P_1(\uparrow)P_3(\downarrow) = \frac{1}{4} && (60) \end{aligned}$$

7. Work through problem 9.1 on page 990 in Cohen-Tannoudji, transcribed below:

Consider a spin 1/2 particle. Call its spin $\vec{S}$, and its orbital angular momentum, $\vec{L}$, and its state vector $|\psi\rangle$. The two functions $\psi_+(\vec{r})$ and $\psi_-(\vec{r})$ are defined by

$$\psi_\pm(\vec{r}) = \langle \vec{r}, \pm|\psi\rangle, \tag{61}$$

where + indicates spin up relative to the z-axis, and − indicates spin down.
Assume that:

$$\psi_+(\vec{r}) = R(r)\left[Y_0^0(\theta,\phi) + \frac{1}{\sqrt{3}}Y_1^0(\theta,\phi)\right] \tag{62}$$

$$\psi_-(\vec{r}) = \frac{R(r)}{\sqrt{3}}\left[Y_1^1(\theta,\phi) - Y_1^0(\theta,\phi)\right] \tag{63}$$

where r, θ, and ϕ are the coordinates of the particle and $R(r)$ is a given function of r.

(a) What condition must $R(r)$ satisfy for $|\psi\rangle$ to be normalized?

The normalization condition is

$$\begin{aligned} 1 &= \int d^3r \left(|\psi_+(\vec{r})|^2 + |\psi_-(\vec{r})|^2\right) \\ &= \int_0^\infty r^2 dr\,|R(r)|^2 \left(1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) \\ &= 2\int_0^\infty r^2 dr\,|R(r)|^2 \end{aligned} \tag{64}$$

so that the condition on $R(r)$ is

$$\int_0^\infty r^2 dr\,|R(r)|^2 = \frac{1}{2} \tag{65}$$

(b) S_z is measured with the particle in state $|\psi\rangle$. What results can be found, and with what probabilities? Same question for L_z, then for S_x.

In Dirac notation, the state of the system in the basis $\{|R,\ell,m_\ell,m_s\rangle\}$, where $|R,\ell,m_\ell,m_s\rangle = |R\rangle^{(r)} \otimes |\ell.m_\ell\rangle^{(\Omega)} \otimes |m_s\rangle^{(s)}$, and $\langle R|R\rangle^{(r)} = \frac{1}{2}$.

$$|\psi\rangle = |R,0,0,+\rangle + \frac{1}{\sqrt{3}}|R,1,0,+\rangle + \frac{1}{\sqrt{3}}|R,1,1,-\rangle - \frac{1}{\sqrt{3}}|R,1,0,-\rangle \tag{66}$$

If S_z is measured, the two possible outcomes are $\pm\hbar/2$. The probability to obtain $\hbar/2$ is

$$\begin{aligned} P_+ &= \langle\psi||+\rangle\langle+|^{(s)}|\psi\rangle \\ &= \left[\langle R,0,0,+| + \frac{1}{\sqrt{3}}\langle R,1,0,+|\right]\left[|R,0,0,+\rangle + \frac{1}{\sqrt{3}}|R,1,0,+\rangle\right] \qquad (67) \\ &= \frac{4}{3}\langle R|R\rangle \\ &= \frac{2}{3} \qquad (68) \end{aligned}$$

The probability to obtain $-\hbar/2$ is then

$$P_- = 1 - P_+ = \frac{1}{3} \tag{69}$$

A measurement of L_z could obtain the results 0, or $\hbar$. The corresponding probabilities are

$$\begin{aligned} P(1) &= \langle\psi||1\rangle\langle 1|^{(\phi)}|\psi\rangle \\ &= \frac{1}{3}\langle R,1,1,-|R,1,1,-\rangle \\ &= \frac{1}{3}\langle R|R\rangle \\ &= \frac{1}{6} \end{aligned} \tag{70}$$

and

$$P(\hbar) = 1 - P(0) = \frac{5}{6} \tag{71}$$

For S_x, the possible results are $\pm\hbar/2$. The corresponding probabilities are

$$\begin{aligned} P(+) &= \langle\psi||\uparrow_x\rangle\langle\uparrow_x|^{(s)}|\psi \\ &= \frac{1}{2}\langle\psi|\left(|+\rangle + |-\rangle\right)\left(\langle+| + \langle-|\right)|\psi\rangle \\ &= \frac{1}{2}\left(\langle R,0,0| + \frac{1}{\sqrt{3}}\langle R,1,1|\right)\left(|R,0,0\rangle + \frac{1}{\sqrt{3}}|R,1,1\rangle\right) \\ &= \frac{2}{3} \end{aligned} \tag{72}$$

$$\begin{aligned} P(-) &= 1 - P(+) \\ &= \frac{1}{3} \end{aligned} \tag{73}$$

(c) A measurement of L^2, with the particle in state $|\psi\rangle$, yielded zero. What state describes the particle just after this measurement? Same question if the measurement of L^2 had given $2\hbar^2$.

A measurement of L^2 can yield only two possible outcomes for this particular state. They are 0, for $\ell = 0$, and $2\hbar^2$, for $\ell = 1$.
The probability to obtain $\ell = 0$ is

$$\begin{aligned} P(0) &= \langle\psi||0\rangle\langle 0|^{(\ell)}|\psi\rangle \\ &= \langle R,0,0,+|R,0,0,+\rangle \\ &= \langle R|R\rangle \\ &= \frac{1}{2} \end{aligned} \tag{74}$$

The state after the measurement is then

$$\begin{aligned} |\psi'\rangle &= \frac{|0\rangle\langle 0|^{(\ell)}|\psi\rangle}{\sqrt{P(0)}} \\ &= \sqrt{2}|R,0,0,+\rangle \end{aligned} \tag{75}$$

The probability to obtain $2\hbar^2$ is

$$P(1) = 1 - P(0) = \frac{1}{2} \tag{76}$$

and the state after obtaining this result is

$$\begin{aligned} |\psi'\rangle &= \frac{|1\rangle\langle 1|^{(\ell)}|\psi\rangle}{\sqrt{P(1)}} \\ &= \sqrt{\frac{2}{3}}\left(|R,1,0,+\rangle + |R,1,1,-\rangle - |R,1,0,-\rangle\right) \end{aligned} \tag{77}$$