PHYS852 Quantum Mechanics II, Spring 2010

HOMEWORK ASSIGNMENT 3

Topics covered: Unitary transformations, translation, rotation, vector operators

1. [25]**Symmetry**: A quantum system is said to posses a 'symmetry' if the Hamiltonian operator, H, is invariant under the associated transformation. In other words, if $H' = H$, where $H' := U^\dagger H U$.
   (a) [5] Show that $H' = H$ is equivalent to $[H, U] = 0$
   (b) [5] Any hermitian operator can be used to generate a unitary operator via $U = e^{-iG\phi}$, where $G^\dagger = G$ is the 'generator' of the symmetry transformation, and ϕ is a free parameter. Show that $[H, G] = 0$ is necessary and sufficient for H to be symmetric under U.
   (c) [5] Show that when $[H, G] = 0$, the probability distribution over the eigenvalues of G does not change in time. In QM this means that G is a 'constant of motion'. Must a QM constant of motion have a well-defined value?
   (d) [5] What operator is the 'generator' of translation? If a system possesses 'translational symmetry' what operator is a constant of motion?
   (e) [5] Consider a particle described by the Hamiltonian

$$H = \frac{P^2}{2M} + V(X). \tag{1}$$

   What operator is the generator of translation? Show that H has translational symmetry only if $V(x) = V_0$.

2. [25] Consider a system described by the Hamiltonian

$$H = \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 + MgX, \tag{2}$$

   where g has units of acceleration.
   (a) [5] Show that $U_T^\dagger(d) X U_T(d) = X + d$ and $U_T^\dagger(d) P U_T(d) = P$.
   (b) [5] Solve for d and E_0 such that $H' := U_T^\dagger(d) H U_T(d)$ satisfies

$$H' = E_0 + \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 \tag{3}$$

   (c) [5] Let $|\phi_n'\rangle$, $n = 0, 1, 2, \ldots$ be the n^{th} eigenstate of H', with corresponding eigenvalue E_n'. What are E_n' and $\phi_n'(x) = \langle x|\phi_n'\rangle$?
   (d) [5] Show that $|\phi_n\rangle := U_T(d)|\phi_n'\rangle$ is an eigenstate of H with eigenvalue E_n. What is the relationship between E_n and E_n'?
   (e) [5] What is the relationship between $\phi_n(x) := \langle x|\phi_n\rangle$ and $\phi_n'(x)$? What is $\phi_n(x)$?

3. [10] Show explicitly that the momentum operator of a particle $\vec{P}$ is a vector operator with respect to rotation. Show that the operator $P^2 = \vec{P} \cdot \vec{P}$ is invariant under rotation about any axis (hint: chose a coordinate system where the axis of rotation is the z-axis).

4. [40/35] Consider an infinitesimal rotation about an arbitrary axis, described by the unitary operator

$$U_R(\vec{\epsilon}) = e^{-\frac{i}{\hbar}\vec{L}\cdot\vec{\epsilon}} = 1 - \frac{i}{\hbar}L_1\epsilon_1 - \frac{i}{\hbar}L_2\epsilon_2 - \frac{i}{\hbar}L_3\epsilon_3. \tag{4}$$

where $\vec{L} = \sum_j L_j\vec{e}_j$ and $\vec{\epsilon} = \sum_j \epsilon_j\vec{e}_j$, with $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ being a right-handed set of orthogonal unit vectors. Using this notation, the angular momentum components are given by $L_j = \sum_{k,\ell}\epsilon_{j,k,\ell}R_kP_\ell$, with $\epsilon_{j,k,\ell}$ being the totally antisymmetric Levi-Cevita tensor,

$$\epsilon_{jk\ell} = \begin{cases} 0; & \text{any index repeated} \\ 1; & \text{cyclic permutations of } \{j,k,\ell\} = \{1,2,3\} \\ -1; & \text{cyclic permutations of } \{j,k,\ell\} = \{3,2,1\} \end{cases} . \tag{5}$$

The components of $\vec{R}$ and $\vec{P}$ satisfy the commutation relation $[R_j, P_k] = i\hbar\delta_{j,k}$.

(a) [10] Evaluate $R_j' = U_R^\dagger(\vec{\epsilon})R_jU_R(\vec{\epsilon})$ for each component of the position operator $\vec{R} = \sum_j R_j\vec{e}_j$, and use this to deduce the 3×3 matrix, $M(\vec{\epsilon})$ that rotates an ordinary vector by the infinitesimal angle $\vec{\epsilon}$.

(b) [5] Show that $M(-\vec{\epsilon}) = M^T(\vec{\epsilon})$, then show that $M^T(\vec{\epsilon}) = M^{-1}(\vec{\epsilon})$ by showing that $M^T(\vec{\epsilon})M(\vec{\epsilon}) = I$.

(c) [5] Now consider a finite rotation by $\vec{\delta} = \sum_j \delta_j\vec{e}_j$, described by the 3×3 matrix $M(\vec{\delta})$. Clearly we must have $M(\vec{\delta}) = M^N(\vec{\delta}/N)$. Take the limit as $N \to \infty$, and use your result to part (a) to show that we can put $M(\vec{\delta})$ into the form:

$$M(\vec{\delta}) = \lim_{N\to\infty}\left(1 - \frac{1}{N}\Lambda(\vec{\delta})\right)^N = e^{-\Lambda(\delta)} \tag{6}$$

where $\Lambda(\vec{\delta})$ is a 3×3 antisymmetric matrix, whose components are given by $\Lambda_{j,k}(\vec{\delta}) = \sum_\ell \epsilon_{j,k,\ell}\delta_\ell$.

(d) [5] Show that the eigenvalues of $\Lambda(\vec{\delta})$ are $\omega_0 = 0$, and $\omega_\pm = \pm i\delta$, where $\delta = |\vec{\delta}|$.

(e) Show that the eigenvectors of $\Lambda(\vec{\delta})$ are

$$\begin{aligned} \vec{u}_0 &= \frac{\vec{\delta}}{\delta} && (7) \\ \vec{u}_\pm &= \frac{(\delta_1\delta_2 \pm i\delta\delta_3)\vec{e}_1 + (\delta_2^2 - \delta^2)\vec{e}_2 + (\delta_2\delta_3 \mp i\delta\delta_1)\vec{e}_3}{\sqrt{2\delta^2(\delta^2 - \delta_2^2)}} && (8) \end{aligned}$$

(f) [5] Based on your result to part (e), show that

$$M(\vec{\delta})\vec{V} = \vec{u}_0(\vec{u}_0\cdot\vec{V}) + \vec{u}_-e^{i\delta}(\vec{u}_+\cdot\vec{V}) + \vec{u}_+e^{-i\delta}(\vec{u}_-\cdot\vec{V}) \tag{9}$$

where $\vec{V}$ is an arbitrary vector.

(g) [5+5 bonus] Based on your results to parts (e) and (f), show that

$$\vec{V}' = U_R^\dagger(\vec{\delta})\vec{V}U_R(\vec{\delta}) = M(\vec{\delta})\vec{V} = \frac{\vec{\delta}(\vec{\delta}\cdot\vec{V})}{\delta^2} + \left[\vec{V} - \frac{\vec{\delta}(\vec{\delta}\cdot\vec{V})}{\delta^2}\right]\cos(\delta) + \frac{\vec{\delta}\times\vec{V}}{\delta}\sin(\delta) \tag{10}$$