

Topics covered: Time-independent perturbation theory.

1. [30] **Two-Level System:** Consider the system described by $H = \delta S_z + \Omega S_x$, with $\delta > 0$, where S_x and S_z are components of the spin vector of an $s = 1/2$ particle. Treat the S_z term as the bare Hamiltonian.

- (a) [15] Use perturbation theory to compute the eigenvalues and eigenvectors of H . Compute all terms up to fourth-order in Ω .

The bare eigenstates are

$$|1^{(0)}\rangle = |\downarrow\rangle \quad (1)$$

$$|2^{(0)}\rangle = |\uparrow\rangle \quad (2)$$

with bare eigenvalues

$$E_1^{(0)} = -\hbar\delta/2 \quad (3)$$

$$E_2^{(0)} = \hbar\delta/2 \quad (4)$$

we therefore need to iterate the following equations up to $j = 4$:

$$E_n^{(j)} = \langle n^{(0)}|V|n^{(j-1)}\rangle - \sum_{k=1}^{j-1} E_n^{(k)} \langle n^{(0)}|n^{(j-k)}\rangle \quad (5)$$

$$|n^{(j)}\rangle = |n^{(0)}\rangle \left[-\frac{1}{2} \sum_{k=1}^{j-1} \langle n^{(k)}|n^{(j-k)}\rangle \right] - \sum_{m \neq n} |m^{(0)}\rangle \left[\frac{\langle m^{(0)}|V|n^{(j-1)}\rangle}{E_{mn}} - \sum_{k=1}^{j-1} \frac{E_n^{(k)}}{E_{mn}} \langle m^{(0)}|n^{(j-k)}\rangle \right] \quad (6)$$

for $j = 1$ we have

$$E_n^{(1)} = \Omega \langle n^{(0)}|S_x|n^{(0)}\rangle = 0 \quad (7)$$

and for the states we have

$$|n^{(1)}\rangle = - \sum_{m \neq n} |m^{(0)}\rangle \frac{\Omega}{E_{mn}} \langle m^{(0)}|S_x|n^{(0)}\rangle = - \sum_{m \neq n} \frac{\hbar\Omega}{2E_{mn}} |m^{(0)}\rangle \quad (8)$$

which gives

$$|1^{(1)}\rangle = -\frac{\Omega}{2\delta} |\uparrow\rangle \quad (9)$$

$$|2^{(1)}\rangle = \frac{\Omega}{2\delta} |\downarrow\rangle \quad (10)$$

for $j = 2$ we have

$$E_n^{(2)} = \Omega \langle n^{(0)}|S_x|n^{(1)}\rangle - E_n^{(1)} = \Omega \langle n^{(0)}|S_x|n^{(1)}\rangle \quad (11)$$

which gives

$$E_1^{(2)} = -\frac{\Omega^2}{2\delta} \langle \downarrow|S_x|\uparrow\rangle = -\frac{\hbar\Omega^2}{4\delta} \quad (12)$$

$$E_2^{(2)} = \frac{\Omega^2}{2\delta} \langle \uparrow | S_x | \downarrow \rangle = \frac{\hbar\Omega^2}{4\delta} \quad (13)$$

and for the states

$$\begin{aligned} |n^{(2)}\rangle &= |n^{(0)}\rangle \left[-\frac{1}{2} \langle n^{(1)} | n^{(1)} \rangle \right] - \sum_{m \neq n} |m^{(0)}\rangle \left[\frac{\Omega}{E_{mn}} \langle m^{(0)} | S_x | n^{(1)} \rangle - \frac{E_n^{(1)}}{E_{mn}} \langle m^{(0)} | n^{(1)} \rangle \right] \\ &= -\frac{\Omega^2}{8\delta^2} |n^{(0)}\rangle \end{aligned} \quad (14)$$

For $j = 3$ we have for the energies

$$E_n^{(3)} = \Omega \langle n^{(0)} | S_x | n^{(2)} \rangle - E_n^{(1)} \langle n^{(0)} | n^{(2)} \rangle - E_n^{(2)} \langle n^{(0)} | n^{(1)} \rangle = 0 \quad (15)$$

and for the states we have

$$\begin{aligned} |n^{(3)}\rangle &= -\frac{1}{2} |n^{(0)}\rangle \left[\langle n^{(1)} | n^{(2)} \rangle + \langle n^{(2)} | n^{(1)} \rangle \right] \\ &\quad - \sum_{m \neq n} |m^{(0)}\rangle \left[\frac{\Omega}{E_{mn}} \langle m^{(0)} | S_x | n^{(2)} \rangle - \frac{E_n^{(1)}}{E_{mn}} \langle m^{(0)} | n^{(2)} \rangle - \frac{E_n^{(2)}}{E_{mn}} \langle m^{(0)} | n^{(1)} \rangle \right] \\ &= \sum_{m \neq n} |m^{(0)}\rangle \left[\frac{\hbar\Omega^3}{16\delta^2 E_{mn}} + \frac{E_n^{(2)}}{E_{mn}} \langle m^{(0)} | n^{(1)} \rangle \right] \end{aligned} \quad (16)$$

which gives

$$|1^{(3)}\rangle = | \uparrow \rangle \left[\frac{\Omega^3}{16\delta^3} + \frac{\Omega^3}{8\delta^3} \right] = \frac{3\Omega^3}{16\delta^3} | \uparrow \rangle \quad (17)$$

$$|2^{(3)}\rangle = | \downarrow \rangle \left[-\frac{\Omega^3}{16\delta^3} - \frac{\Omega^3}{8\delta^3} \right] = -\frac{3\Omega^3}{16\delta^3} | \downarrow \rangle \quad (18)$$

And finally, for $j = 4$, we have

$$\begin{aligned} E_n^{(4)} &= \Omega \langle n^{(0)} | S_x | n^{(3)} \rangle - E_n^{(1)} \langle n^{(0)} | n^{(3)} \rangle - E_n^{(2)} \langle n^{(0)} | n^{(2)} \rangle - E_n^{(3)} \langle n^{(0)} | n^{(1)} \rangle \\ &= \Omega \langle n^{(0)} | S_x | n^{(3)} \rangle + \frac{\Omega^2 E_n^{(2)}}{8\delta^2} \end{aligned} \quad (19)$$

which gives

$$E_1^{(4)} = \frac{3\hbar\Omega^4}{32\delta^3} - \frac{\hbar\Omega^4}{32\delta^3} = \frac{\hbar\Omega^4}{16\delta^3} \quad (20)$$

$$E_2^{(4)} = -\frac{3\hbar\Omega^4}{32\delta^3} + \frac{\hbar\Omega^4}{32\delta^3} = -\frac{\hbar\Omega^4}{16\delta^3} \quad (21)$$

and for the states, we have

$$\begin{aligned} |n^{(4)}\rangle &= -\frac{1}{2} |n^{(0)}\rangle \left[\langle n^{(1)} | n^{(3)} \rangle + \langle n^{(2)} | n^{(2)} \rangle + \langle n^{(3)} | n^{(1)} \rangle \right] \\ &\quad - \sum_{m \neq n} |m^{(0)}\rangle \left[\frac{\Omega}{E_{mn}} \langle m^{(0)} | S_x | n^{(3)} \rangle - \frac{E_n^{(1)}}{E_{mn}} \langle m^{(0)} | n^{(3)} \rangle - \frac{E_n^{(2)}}{E_{mn}} \langle m^{(0)} | n^{(2)} \rangle - \frac{E_n^{(3)}}{E_{mn}} \langle m^{(0)} | n^{(1)} \rangle \right] \\ &= |n^{(0)}\rangle \left[-\langle n^{(1)} | n^{(3)} \rangle - \frac{\Omega^4}{128\delta^4} \right] \end{aligned} \quad (22)$$

which gives

$$|1^{(4)}\rangle = |\downarrow\rangle \left[\frac{3\Omega^4}{32\delta^4} - \frac{\Omega^4}{128\delta^4} \right] = \frac{11\Omega^4}{128\delta^4} |\downarrow\rangle \quad (23)$$

$$|2^{(4)}\rangle = |\uparrow\rangle \left[\frac{3\Omega^4}{32\delta^4} - \frac{\Omega^4}{128\delta^4} \right] = \frac{11\Omega^4}{128\delta^4} |\uparrow\rangle \quad (24)$$

Putting the pieces together gives

$$E_1 \approx -\frac{\hbar\delta}{2} - \frac{\hbar\Omega^2}{4\delta} + \frac{\hbar\Omega^4}{16\delta^3} \quad (25)$$

$$E_2 \approx \frac{\hbar\delta}{2} + \frac{\hbar\Omega^2}{4\delta} - \frac{\hbar\Omega^4}{16\delta^3} \quad (26)$$

and

$$|1\rangle \approx |\downarrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right] + |\uparrow\rangle \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3} \right] \quad (27)$$

$$|2\rangle \approx |\downarrow\rangle \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3} \right] + |\uparrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right] \quad (28)$$

- (b) [5] Expand the exact eigenvalues and eigenvectors around $\Omega = 0$ and compare to the perturbation theory results.

The exact results are

$$E_1 = -\frac{\hbar}{2} \sqrt{\delta^2 + \Omega^2} \approx -\frac{\hbar\delta}{2} - \frac{\hbar\Omega^2}{4\delta} + \frac{\hbar\Omega^4}{16\delta^3} \quad (29)$$

$$E_2 = \frac{\hbar}{2} \sqrt{\delta^2 + \Omega^2} \approx \frac{\hbar\delta}{2} + \frac{\hbar\Omega^2}{4\delta} - \frac{\hbar\Omega^4}{16\delta^3} \quad (30)$$

$$|1\rangle = \frac{(\delta + \sqrt{\delta^2 + \Omega^2})|\downarrow\rangle - \Omega|\uparrow\rangle}{\sqrt{(\delta + \sqrt{\delta^2 + \Omega^2})^2 + \Omega^2}} \approx |\downarrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right] + |\uparrow\rangle \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3} \right] \quad (31)$$

$$|2\rangle = \frac{\omega|\downarrow\rangle + (\delta + \sqrt{\delta^2 + \Omega^2})|\uparrow\rangle}{\sqrt{\Omega^2 + (\delta + \sqrt{\delta^2 + \Omega^2})^2}} \approx |\downarrow\rangle \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3} \right] + |\uparrow\rangle \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right] \quad (32)$$

- (c) [10] Verify that the states computed in (a) are normalized to unity and orthogonal up to fourth-order.

$$\begin{aligned} \langle 1|1\rangle &= \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right]^2 + \left[-\frac{\Omega}{2\delta} + \frac{3\Omega^3}{16\delta^3} \right]^2 + O(\Omega^5) \\ &= 1 - \frac{\Omega^2}{4\delta^2} + \frac{11\Omega^4}{64\delta^4} + \frac{\Omega^4}{64\delta^4} + \frac{\Omega^2}{4\delta^2} - \frac{3\Omega^4}{16\delta^4} + O(\Omega^5) \\ &= 1 + O(\Omega^5) \end{aligned} \quad (33)$$

$$\begin{aligned} \langle 2|2\rangle &= \left[\frac{\Omega}{2\delta} - \frac{3\Omega^3}{16\delta^3} \right]^2 + \left[1 - \frac{\Omega^2}{8\delta^2} + \frac{11\Omega^4}{128\delta^4} \right]^2 + O(\Omega^5) \\ &= \frac{\Omega^2}{4\delta^2} - \frac{3\Omega^4}{16\delta^4} + 1 - \frac{\Omega^2}{4\delta^2} + \frac{11\Omega^4}{64\delta^4} + \frac{\Omega^4}{64\delta^4} + O(\Omega^5) \\ &= 1 + O(\Omega^5) \end{aligned} \quad (34)$$

It is clear by inspection of Eqs. (27) and (28) that $\langle 1|2\rangle = 0$.

2. [20] **Resonance-frequency shifts:** Consider a system with a 3-dimensional Hilbert space spanned by states $|a\rangle$, $|b\rangle$, and $|c\rangle$. In the basis $\{|a\rangle, |b\rangle, |c\rangle\}$, let the bare Hamiltonian of the system be

$$H_0 = \hbar\Delta \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{pmatrix}. \quad (35)$$

For the case where the system is perturbed by the operator

$$V = \hbar\chi \begin{pmatrix} 1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad (36)$$

also given in $\{|a\rangle, |b\rangle, |c\rangle\}$ basis. Calculate the *shifts* in the *resonance frequencies* of the full system relative to those of the unperturbed system, to second-order in χ .

The eigenvalues of H_0 are the solutions to $\det |H_0 - \omega| = 0$, which yields the characteristic equation

$$(1 - \omega) [(1 - \omega)(3 - \omega) - 1] - [-1 + 3 - \omega] + [1 - (1 - \omega)] = 0 \quad (37)$$

which simplifies to

$$(1 - \omega)\omega(\omega - 4) = 0 \quad (38)$$

so that the eigenvalues of H_0 are $E_1^{(0)} = 0$, $E_2^{(0)} = \hbar\Delta$, and $E_3^{(0)} = 4\hbar\Delta$.

The three bare resonance frequencies are therefore

$$\omega_1^{(0)} = \frac{E_2^{(0)} - E_1^{(0)}}{\hbar} = \Delta, \quad (39)$$

$$\omega_2^{(0)} = \frac{E_3^{(0)} - E_2^{(0)}}{\hbar} = 3\Delta, \quad (40)$$

and

$$\omega_3^{(0)} = \frac{E_3^{(0)} - E_1^{(0)}}{\hbar} = 4\Delta. \quad (41)$$

The eigenvectors satisfy

$$\begin{pmatrix} 1 - \frac{E_n^{(0)}}{\hbar\Delta} & -1 & 1 \\ -1 & 1 - \frac{E_n^{(0)}}{\hbar\Delta} & -1 \\ 1 & -1 & 3 - \frac{E_n^{(0)}}{\hbar\Delta} \end{pmatrix} \begin{pmatrix} \langle a|n^{(0)}\rangle \\ \langle b|n^{(0)}\rangle \\ \langle c|n^{(0)}\rangle \end{pmatrix} = 0 \quad (42)$$

taking $\langle a|n^{(0)}\rangle = 1$ gives for $n = 1$,

$$-\langle b|1^{(0)}\rangle + \langle c|1^{(0)}\rangle = -1 \quad (43)$$

$$-\langle b|1^{(0)}\rangle + 3\langle c|1^{(0)}\rangle = -1 \quad (44)$$

which has the solution $\langle b|1^{(0)}\rangle = 1$ and $\langle c|1^{(0)}\rangle = 0$, so that

$$|1^{(0)}\rangle = \frac{1}{\sqrt{2}} [|a\rangle + |b\rangle] \quad (45)$$

for $n = 2$ the equations are

$$-\langle b|2^{(0)}\rangle + \langle c|2^{(0)}\rangle = 0 \quad (46)$$

$$-\langle c|2^{(0)}\rangle = 1 \quad (47)$$

so we see that

$$|2^{(0)}\rangle = \frac{1}{\sqrt{3}} [|a\rangle - |b\rangle - |c\rangle] \quad (48)$$

and for $n = 3$, we have

$$-\langle b|3^{(0)}\rangle + \langle c|3^{(0)}\rangle = 3 \quad (49)$$

$$-3\langle b|3^{(0)}\rangle - \langle c|3^{(0)}\rangle = 1 \quad (50)$$

which gives

$$|3^{(0)}\rangle = \frac{1}{\sqrt{6}} [|a\rangle - |b\rangle + 2|c\rangle] \quad (51)$$

Switching to the bare-eigenstate basis, we have

$$H_0 = \hbar\Delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad (52)$$

and

$$V = \begin{pmatrix} \langle 1^{(0)}|V|1^{(0)}\rangle & \langle 1^{(0)}|V|2^{(0)}\rangle & \langle 1^{(0)}|V|3^{(0)}\rangle \\ \langle 2^{(0)}|V|1^{(0)}\rangle & \langle 2^{(0)}|V|2^{(0)}\rangle & \langle 2^{(0)}|V|3^{(0)}\rangle \\ \langle 3^{(0)}|V|1^{(0)}\rangle & \langle 3^{(0)}|V|2^{(0)}\rangle & \langle 3^{(0)}|V|3^{(0)}\rangle \end{pmatrix} \quad (53)$$

we have

$$V|1^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{2}} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \quad (54)$$

$$V|2^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{3}} \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} \quad (55)$$

and

$$V|3^{(0)}\rangle = \frac{\hbar\chi}{\sqrt{6}} \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} \quad (56)$$

from which we can obtain

$$V = \sqrt{\frac{3}{2}}\hbar\chi \begin{pmatrix} 0 & 1 & \sqrt{2} \\ 1 & 0 & -\sqrt{3} \\ \sqrt{2} & -\sqrt{3} & 0 \end{pmatrix} \quad (57)$$

The first-order energy shifts are zero, while the second-order terms are given by

$$E_1^{(2)} = -\frac{|V_{21}|^2}{\hbar\Delta} - \frac{|V_{31}|^2}{4\hbar\Delta} = -\frac{3\hbar\chi^2}{2\Delta} - \frac{6\hbar\chi^2}{8\Delta} = -\frac{9\hbar\chi^2}{4\Delta} \quad (58)$$

$$E_2^{(2)} = \frac{|V_{12}|^2}{\hbar\Delta} - \frac{|V_{32}|^2}{3\hbar\Delta} = \frac{3\hbar\chi^2}{2\Delta} - \frac{9\hbar\chi^2}{6\Delta} = 0 \quad (59)$$

$$E_3^{(2)} = \frac{|V_{13}|^2}{4\hbar\Delta} + \frac{|V_{23}|^2}{3\hbar\Delta} = \frac{6\hbar\chi^2}{8\Delta} + \frac{9\hbar\chi^2}{6\Delta} = \frac{9\hbar\chi^2}{4\Delta} \quad (60)$$

Thus the resonance frequency shifts are therefore

$$\omega_1^{(3)} = \frac{E_2^{(2)} - E_1^{(2)}}{\hbar} = \frac{9\chi^2}{4\Delta}, \quad (61)$$

$$\omega_2^{(2)} = \frac{E_3^{(2)} - E_2^{(2)}}{\hbar} = \frac{9\chi^2}{4\Delta}, \quad (62)$$

and

$$\omega_3 = \frac{E_3^{(2)} - E_1^{(2)}}{\hbar} = \frac{9\chi^2}{2\Delta}. \quad (63)$$

3. [15] Consider a pair of quantum harmonic oscillators, described by the bare Hamiltonian

$$H_0 = \hbar\omega(A^\dagger A + 1/2) + \hbar\Omega(B^\dagger B + 1/2).$$

Assume that $\omega < \Omega < 2\omega$, and determine the three lowest bare energy eigenvalues and eigenvectors. Consider the perturbation

$$V = \hbar g (A^\dagger A^\dagger B + B^\dagger A A).$$

Show that two of the three lowest levels are exact eigenstates of $H = H_0 + V$. For the remaining bare level, compute the first non-vanishing corrections to the eigenvalue and eigenvector.

The three lowest energy levels are $|1^{(0)}\rangle = |00\rangle$ with energy $E_1^{(0)} = \hbar\frac{\omega+\Omega}{2}$, $|2^{(0)}\rangle = |10\rangle$, with energy $E_2^{(0)} = \hbar\frac{3\omega+\Omega}{2}$, and $|3^{(0)}\rangle = |01\rangle$ with energy $E_3^{(0)} = \hbar\frac{\omega+3\Omega}{2}$.

We have $V|1^{(0)}\rangle = 0$, and $V|2^{(0)}\rangle = 0$ so that $|1^{(0)}\rangle$ and $|2^{(0)}\rangle$ are eigenstates of V with eigenvalue 0. For level $|3\rangle$ we have $V|3^{(0)}\rangle = \hbar\sqrt{2}g|20\rangle$. This means that

$$E_3^{(1)} = 0 \tag{64}$$

so that

$$E_3^{(2)} = -\frac{2\hbar g^2}{(2\omega - \Omega)} \tag{65}$$

and

$$|3^{(1)}\rangle = -|20\rangle \frac{\sqrt{2}g}{(2\omega - \Omega)} \tag{66}$$

are the first non-vanishing corrections.

4. [15] Consider a particle of mass M confined to a 1-dimensional box of length L . Use perturbation theory to calculate the effects of adding a tilt to the box, represented by adding the linear potential

$$V_{\text{tilt}}(x) = \hbar\beta \left(\frac{x}{L} - \frac{1}{2} \right)$$

to the box potential,

$$V_{\text{box}}(x) = \begin{cases} 0; & 0 < x < L \\ \infty; & \text{else} \end{cases}$$

Calculate the three lowest perturbed eigenstates to first-order and their corresponding eigenvalues to second-order.

We have

$$E_1^{(0)} = \frac{\hbar^2 \pi^2}{2ML^2} \quad (67)$$

$$E_2^{(0)} = \frac{4\hbar^2 \pi^2}{2ML^2} \quad (68)$$

and

$$E_3^{(0)} = \frac{9\hbar^2 \pi^2}{2ML^2} \quad (69)$$

As well as

$$\langle x | 1^{(0)} \rangle = \sqrt{\frac{2}{L}} \sin(\pi x/L) \quad (70)$$

$$\langle x | 2^{(0)} \rangle = \sqrt{\frac{2}{L}} \sin(2\pi x/L) \quad (71)$$

and

$$\langle x | 3^{(0)} \rangle = \sqrt{\frac{2}{L}} \sin(3\pi x/L) \quad (72)$$

The first-order energy shifts are all zero because $V_{\text{tilt}}(x)$ is odd with respect to the center of the box. We will need the matrix elements

$$\begin{aligned} V_{mn} &= \frac{2\hbar\beta}{L} \int_0^L dx \sin(m\pi x/L) \left(\frac{x}{L} - \frac{1}{2} \right) \sin(n\pi x/L) \\ &= 2\hbar\beta \int_0^1 dy \sin(m\pi y) \left(y - \frac{1}{2} \right) \sin(n\pi y) \\ &= \frac{4\hbar\beta mn ((-1)^{m+n} - 1)}{(m^2 - n^2)^2 \pi^2} \end{aligned} \quad (73)$$

The first-order states are

$$|1^{(1)}\rangle = - \sum_{n=2}^{\infty} |n^{(0)}\rangle \frac{V_{n1}}{E_n^{(0)} - E_1^{(0)}} = \frac{32\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |2m^{(0)}\rangle \frac{m}{((2m)^2 - 1)^3} \quad (74)$$

$$|2^{(1)}\rangle = - \sum_{n \neq 2} |n^{(0)}\rangle \frac{V_{n2}}{E_n^{(0)} - E_2^{(0)}} = \frac{32\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |m^{(0)}\rangle \frac{2m-1}{((2m-1)^2 - 4)^3} \quad (75)$$

$$|3^{(0)}\rangle = - \sum_{n \neq 3} |n^{(0)}\rangle \frac{V_{n3}}{E_n^{(0)} - E_3^{(0)}} = \frac{96\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |m^{(0)}\rangle \frac{m}{(2m)^2 - 9)^3} \quad (76)$$

and the corresponding first-order energies are

$$E_1^{(2)} = \langle 1^{(0)} | V | 1^{(1)} \rangle = -\frac{128\beta^2 ML^2}{\pi^6} \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)^2 - 1)^5} = -\frac{(15 - \pi^2)\beta^2 ML^2}{24\pi^4} \quad (77)$$

$$E_2^{(2)} = \langle 2^{(0)} | V | 2^{(1)} \rangle = -\frac{512\beta^2 ML^2}{\pi^6} \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)^2 - 4)^5} = \frac{(4\pi^2 - 15)\beta^2 ML^2}{384\pi^4} \quad (78)$$

$$E_3^{(2)} = \langle 3^{(0)} | V | 3^{(1)} \rangle = -\frac{1152\beta^2 ML^2}{\pi^6} \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)^2 - 9)^5} = \frac{(9\pi^2 - 15)\beta^2 ML^2}{1944\pi^4} \quad (79)$$

The results are then

$$E_1 \approx \frac{\pi^2 \hbar^2}{2ML^2} - \frac{(15 - \pi^2)\beta^2 ML^2}{24\pi^4} \quad (80)$$

$$E_2 \approx \frac{4\pi^2 \hbar^2}{2ML^2} + \frac{(4\pi^2 - 15)\beta^2 ML^2}{384\pi^4} \quad (81)$$

$$E_3 \approx \frac{9\pi^2 \hbar^2}{2ML^2} + \frac{(9\pi^2 - 15)\beta^2 ML^2}{1944\pi^4} \quad (82)$$

$$|1\rangle \approx |1^{(0)}\rangle + \frac{32\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |2m^{(0)}\rangle \frac{m}{((2m)^2 - 1)^3} \quad (83)$$

$$|2\rangle \approx |2^{(0)}\rangle + \frac{32\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |m^{(0)}\rangle \frac{2m-1}{((2m-1)^2 - 4)^3} \quad (84)$$

$$|3\rangle \approx |3^{(0)}\rangle + \frac{96\beta ML^2}{\hbar\pi^4} \sum_{m=1}^{\infty} |m^{(0)}\rangle \frac{m}{(2m)^2 - 9)^3} \quad (85)$$