

PHYS852 Quantum Mechanics II, Spring 2010
 HOMEWORK ASSIGNMENT 7: Solutions

Topics covered: addition of three angular momenta, degenerate perturbation theory

1. Consider a system of two spin-1/2 particles, described by \vec{S}_1 and \vec{S}_2 , and one spin-1 particle, described by \vec{S}_3 . Let $\vec{S}' = \vec{S}_1 + \vec{S}_2$. What are the allowed values of s' ? For each allowed value, give the allowed m' values. Use Clebsch Gordan coefficients to express the $|s'm'm_3\rangle$ states in terms of the $|m_1m_2m_3\rangle$ states.

Let $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$. What are the allowed values of the quantum number s ? For each s value, list the allowed m -values. Express the states $|s', s, m\rangle$ as linear superpositions of the $|s'm'm_3\rangle$ states, and then as linear superpositions of the $|m_1m_2m_3\rangle$ states.

Notation:

For $s = 2$, we use $|\uparrow\uparrow\rangle$ for $m = 2$, $|\uparrow\rangle$ for $m = 1$, $|0\rangle$ for $m = 0$, $|\downarrow\rangle$ for $m = -1$, and $|\downarrow\downarrow\rangle$ for $m = -2$;

For $s = 1$, we use $|\uparrow\rangle$ for $m = 1$, $|0\rangle$ for $m = 0$, and $|\downarrow\rangle$ for $m = -1$;

For $s = 1/2$, we use $|\uparrow\rangle$ for $m = 1/2$, and $|\downarrow\rangle$ for $m = -1/2$.

The allowed values of s' are 0, 1

For $s' = 0$, the only allowed value for m' is 0.

For $s' = 1$, the allowed values for m' are $-1, 0, 1$.

The $|s'm'm_3\rangle$ states follow the standard singlet/triplet forms:

$$|s'm'm_3\rangle = |m_1m_2m_3\rangle \tag{1}$$

$$|1\uparrow m_3\rangle = |\uparrow\uparrow m_3\rangle \tag{2}$$

$$|10m_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow m_3\rangle + |\downarrow\uparrow m_3\rangle) \tag{3}$$

$$|1\downarrow m_3\rangle = |\downarrow\downarrow m_3\rangle \tag{4}$$

$$|00m_3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow m_3\rangle - |\downarrow\uparrow m_3\rangle) \tag{5}$$

For $s' = 0$, the allowed value of s is 1.

For $s' = 1$, the allowed values of s are 0, 1, 2.

For $s = 0$, the allowed value of m is 0.

For $s = 1$, the allowed values of m are $-1, 0, 1$.

For $s = 2$, the allowed values of m are $-2, -1, 0, 1, 2$.

$$|s' sm\rangle = |s' m' m_3\rangle = |m_1 m_2 m_3\rangle \quad (6)$$

$$|12 \uparrow\rangle = |1 \uparrow \uparrow\rangle = |\uparrow \uparrow \uparrow\rangle \quad (7)$$

$$|12 \uparrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow 0\rangle + |10 \uparrow\rangle) = \frac{1}{2}(\sqrt{2}|\uparrow \uparrow 0\rangle + |\uparrow \downarrow \uparrow\rangle + |\downarrow \uparrow \uparrow\rangle) \quad (8)$$

$$|120\rangle = \frac{1}{\sqrt{6}}(|1 \uparrow \downarrow\rangle + 2|100\rangle + |1 \downarrow \uparrow\rangle) = \frac{1}{\sqrt{6}}(|\uparrow \uparrow \downarrow\rangle + \sqrt{2}|\uparrow \downarrow 0\rangle + \sqrt{2}|\downarrow \uparrow 0\rangle + |\downarrow \downarrow \uparrow\rangle) \quad (9)$$

$$|12 \downarrow\rangle = \frac{1}{\sqrt{2}}(|10 \downarrow\rangle + |1 \downarrow 0\rangle) = \frac{1}{2}(|\uparrow \downarrow \downarrow\rangle + |\downarrow \downarrow \downarrow\rangle + \sqrt{2}|\downarrow \downarrow 0\rangle) \quad (10)$$

$$|12 \downarrow\rangle = |1 \downarrow \downarrow\rangle = |\downarrow \downarrow \downarrow\rangle \quad (11)$$

$$|11 \uparrow\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow 0\rangle - |10 \uparrow\rangle) = \frac{1}{2}(\sqrt{2}|\uparrow \uparrow 0\rangle - |\uparrow \downarrow \uparrow\rangle - |\downarrow \uparrow \uparrow\rangle) \quad (12)$$

$$|110\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow \downarrow\rangle - |1 \downarrow \uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow \uparrow \downarrow\rangle - |\downarrow \downarrow \uparrow\rangle) \quad (13)$$

$$|11 \downarrow\rangle = \frac{1}{\sqrt{2}}(|10 \downarrow\rangle - |1 \downarrow 0\rangle) = \frac{1}{2}(|\uparrow \downarrow \downarrow\rangle + |\downarrow \downarrow \downarrow\rangle - \sqrt{2}|\downarrow \downarrow 0\rangle) \quad (14)$$

$$|100\rangle = \frac{1}{\sqrt{3}}(|1 \uparrow \downarrow\rangle - |100\rangle + |1 \downarrow \uparrow\rangle) = \frac{1}{\sqrt{6}}(\sqrt{2}|\uparrow \uparrow \downarrow\rangle - |\uparrow \downarrow 0\rangle - |\downarrow \uparrow 0\rangle + \sqrt{2}|\downarrow \downarrow \uparrow\rangle) \quad (15)$$

$$|01 \uparrow\rangle = |00 \uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \uparrow\rangle - |\downarrow \uparrow \uparrow\rangle) \quad (16)$$

$$|010\rangle = |000\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow 0\rangle - |\downarrow \uparrow 0\rangle) \quad (17)$$

$$|01 \downarrow\rangle = |00 \downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \downarrow\rangle - |\downarrow \uparrow \downarrow\rangle) \quad (18)$$

Calculations:

$$S_-|12 \uparrow\rangle = (S'_- + S_{3-})|1 \uparrow \uparrow\rangle = (S_{1-} + S_{2-} + S_{3-})| \uparrow \uparrow \uparrow\rangle \quad (19)$$

$$\sqrt{2 \cdot 3 - 2 \cdot \uparrow}|12 \uparrow\rangle = \sqrt{1 \cdot 2 - 1 \cdot 0}|10 \uparrow\rangle + \sqrt{1 \cdot 2 - 1 \cdot 0}|1 \uparrow 0\rangle = | \downarrow \uparrow \uparrow\rangle + | \uparrow \downarrow \uparrow\rangle + \sqrt{1 \cdot 2 - 1 \cdot 0}| \uparrow \uparrow 0\rangle \quad (20)$$

$$2|12 \uparrow\rangle = \sqrt{2}(|10 \uparrow\rangle + |1 \uparrow 0\rangle) = | \downarrow \uparrow \uparrow\rangle + | \uparrow \downarrow \uparrow\rangle + \sqrt{2}| \uparrow \uparrow 0\rangle \quad (21)$$

$$|12 \uparrow\rangle = \frac{1}{\sqrt{2}}(|10 \uparrow\rangle + |1 \uparrow 0\rangle) = \frac{1}{2}(| \downarrow \uparrow \uparrow\rangle + | \uparrow \downarrow \uparrow\rangle + \sqrt{2}| \uparrow \uparrow 0\rangle). \quad (22)$$

$$S_-|12 \uparrow\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|10 \uparrow\rangle + |1 \uparrow 0\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{2}(| \downarrow \uparrow \uparrow\rangle + | \uparrow \downarrow \uparrow\rangle + \sqrt{2}| \uparrow \uparrow 0\rangle) \quad (23)$$

$$\sqrt{6}|120\rangle = |1 \downarrow \uparrow\rangle + 2|100\rangle + |10 \downarrow\rangle = (| \downarrow \downarrow \uparrow\rangle + \sqrt{2}| \downarrow \uparrow 0\rangle + \sqrt{2}| \uparrow \downarrow 0\rangle + | \uparrow \uparrow \downarrow\rangle) \quad (24)$$

$$|120\rangle = \frac{1}{\sqrt{6}}(|1 \downarrow \uparrow\rangle + 2|100\rangle + |10 \downarrow\rangle) = \frac{1}{\sqrt{6}}(| \downarrow \downarrow \uparrow\rangle + \sqrt{2}| \downarrow \uparrow 0\rangle + \sqrt{2}| \uparrow \downarrow 0\rangle + | \uparrow \uparrow \downarrow\rangle) \quad (25)$$

$$S_-|11 \uparrow\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|1 \uparrow 0\rangle - |10 \uparrow\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{2}(\sqrt{2}| \uparrow \uparrow 0\rangle - | \uparrow \downarrow \uparrow\rangle - | \downarrow \uparrow \uparrow\rangle) \quad (26)$$

$$\sqrt{2}|110\rangle = |1 \uparrow \downarrow\rangle - |1 \downarrow \uparrow\rangle = | \uparrow \uparrow \downarrow\rangle - | \downarrow \downarrow \uparrow\rangle \quad (27)$$

$$|110\rangle = \frac{1}{\sqrt{2}}(|1 \uparrow \downarrow\rangle - |1 \downarrow \uparrow\rangle) = \frac{1}{\sqrt{2}}(| \uparrow \uparrow \downarrow\rangle - | \downarrow \downarrow \uparrow\rangle) \quad (28)$$

$$S_-|110\rangle = (S'_- + S_{3-})\frac{1}{\sqrt{2}}(|1 \uparrow \downarrow\rangle - |1 \downarrow \uparrow\rangle) = (S_{1-} + S_{2-} + S_{3-})\frac{1}{\sqrt{2}}(| \uparrow \uparrow \downarrow\rangle - | \downarrow \downarrow \uparrow\rangle) \quad (29)$$

$$\sqrt{2}|11 \downarrow\rangle = |10 \downarrow\rangle - |1 \downarrow 0\rangle = \frac{1}{\sqrt{2}}(| \downarrow \uparrow \downarrow\rangle + | \uparrow \downarrow \uparrow\rangle - \sqrt{2}| \downarrow \downarrow 0\rangle) \quad (30)$$

$$|11 \downarrow\rangle = \frac{1}{\sqrt{2}}(|10 \downarrow\rangle - |1 \downarrow 0\rangle) = \frac{1}{2}(| \downarrow \uparrow \downarrow\rangle + | \uparrow \downarrow \uparrow\rangle - \sqrt{2}| \downarrow \downarrow 0\rangle) \quad (31)$$

2. Derive general expressions for the energy-shifts up to third-order and state vectors up to second-order in degenerate perturbation theory.

As described in the lecture notes, solving the first- and second-order equations gives:

$$E_{nm} = E_n^{(0)} + \lambda v_{nm} - \lambda^2 \sum_{\substack{n' \neq n \\ m'}} \frac{V_{nmn'm'} V_{n'm'nm}}{\Delta_{n'n}} \quad (32)$$

$$\begin{aligned} |nm\rangle &= |nm^{(0)}\rangle \left[1 - \frac{\lambda^2}{2} \sum_{\substack{n' \neq n \\ m'}} \frac{V_{nmn'm'}}{\Delta_{n'n}} \left(\sum_{m'' \neq m} \sum_{\substack{n''' \neq n \\ m'''}} \frac{V_{n'm'nm''} V_{nm''n''m''} V_{n''m''nm}}{v_{nm''m}^2 \Delta_{n''n}} + V_{n'm'nm} \right) \right] \\ &+ \sum_{m' \neq m} |nm'^{(0)}\rangle \left[\lambda \sum_{\substack{n'' \neq n \\ m''}} \frac{V_{nm'n''m''} V_{n''m''nm}}{v_{nm''m} \Delta_{n''n}} \right] + \sum_{\substack{n' \neq n \\ m'}} |n'm'^{(0)}\rangle \left[-\lambda \frac{V_{n'm'nm}}{\Delta_{n'n}} \left(1 + \lambda \frac{v_{nm}}{\Delta_{n'n}} \right) \right] \\ &+ \lambda^2 \sum_{\substack{n'' \neq n \\ m''}} \left(V_{n'm'n''m''} - \sum_{m''' \neq m} \frac{V_{n'm'nm''} V_{nm''n''m''}}{v_{nm''m}} \right) \frac{V_{n''m''nm}}{\Delta_{n'n} \Delta_{n''n}} \end{aligned} \quad (33)$$

To proceed, we need to use the third-order equations to find the third-order energy-shift and the second-order correction to the $|nm^{(0)}\rangle$ states.

The third-order equation is:

$$(H_0 - E_n^{(0)})|nm^{(3)}\rangle = -V|nm^{(2)}\rangle + E_{nm}^{(3)}|nm^{(0)}\rangle + E_{nm}^{(2)}|nm^{(1)}\rangle + E_{nm}^{(1)}|nm^{(2)}\rangle \quad (34)$$

Hitting from the left with $\langle nm^{(0)}|$ and solving for $E_{nm}^{(3)}$ gives

$$\begin{aligned} E_{nm}^{(3)} &= \langle nm^{(0)}|V|nm^{(2)}\rangle - E_{nm}^{(2)}\langle nm^{(0)}|nm^{(1)}\rangle - E_{nm}^{(1)}\langle nm^{(0)}|nm^{(2)}\rangle \\ &= \sum_{\substack{n' \neq n \\ m'}} V_{nmn'm'} \langle n'm'^{(0)}|nm^{(2)}\rangle \\ &= \sum_{\substack{n' \neq n \\ m'}} \sum_{\substack{n'' \neq n, n' \\ m''}} \frac{V_{nmn'm'} V_{n'm'n''m''} V_{n''m''nm}}{\Delta_{n'n} \Delta_{n''n}} \\ &+ \sum_{\substack{n' \neq n \\ m'}} \frac{V_{nmn'm'} v_{n'm'} V_{n'm'nm}}{\Delta_{n'n}^2} - \sum_{\substack{n' \neq n \\ m'}} \frac{V_{nmn'm'} V_{n'm'nm} v_{nm}}{\Delta_{n'n}^2} \\ &- \sum_{\substack{n' \neq n \\ m'}} \sum_{\substack{n'' \neq n \\ m''}} \sum_{m''' \neq m} \frac{V_{nmn'm'} V_{n'm'nm''} V_{nm''n''m''} V_{n''m''nm}}{v_{nm''m} \Delta_{n'n} \Delta_{n''n}} \end{aligned} \quad (35)$$

Hitting from the left with $\langle nm^{(0)} |$ and solving for $\langle nm^{(0)} | nm^{(2)} \rangle$ gives

$$\begin{aligned}
\langle nm^{(0)} | nm^{(2)} \rangle &= \frac{\langle nm^{(0)} | V | nm^{(2)} \rangle}{v_{nm}} - \frac{E_{nm}^{(2)}}{v_{nm}} \langle nm^{(0)} | nm^{(1)} \rangle \\
&= - \sum_{\substack{n'' \neq n \\ m''}} \frac{V_{nm'n''m''} V_{n''m''nm}}{\Delta_{n''n}^2} \\
&\quad + \sum_{\substack{n'' \neq n \\ m''}} \left[\sum_{\substack{n''' \neq n, n'' \\ m'''}} \frac{V_{nm'n''m''} V_{n''m''n'''m'''} V_{n'''m'''nm}}{v_{nm} \Delta_{n''n} \Delta_{n'''n}} + \frac{V_{nm'n''m''} v_{n''m''} V_{n''m''nm}}{v_{nm} \Delta_{n''n}^2} \right] \\
&\quad - \sum_{\substack{n'' \neq n \\ m''}} \sum_{\substack{n''' \neq n \\ m'''}} \sum_{m'''' \neq m} \frac{V_{nm'n''m''} V_{n''m''nm''''} V_{nm''''n'''m'''} V_{n'''m'''nm}}{v_{nm} v_{nm''''m} \Delta_{n''n} \Delta_{n'''n}} \\
&\quad - \sum_{\substack{n'' \neq n \\ m''}} \sum_{\substack{n''' \neq n \\ m'''}} \frac{V_{nm'n''m''} V_{n''m''nm} V_{nmn'''m'''} V_{n'''m'''nm}}{v_{nm} \Delta_{n''n} v_{nm'm} \Delta_{n'''n}} \tag{36}
\end{aligned}$$

3. Consider a system described in the basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ by the bare Hamiltonian

$$H_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (37)$$

Let the system be perturbed by the operator

$$V = \hbar g \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix} \quad (38)$$

First, determine the ‘good basis’ for degenerate perturbation theory, then compute the eigenvalues and eigenvectors of $H = H_0 + V$ to second-order in perturbation theory.

The bare Hamiltonian has two degenerate subspaces. The set $\{|1\rangle, |2\rangle\}$ spans the space corresponding to $E_1^{(0)} = \hbar\omega_0$, and the set $\{|3\rangle, |4\rangle\}$ spans the space corresponding to $E_2^{(0)} = 2\hbar\omega_0$. By inspection, we see that

$$V_1 = V_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad (39)$$

Therefore the good basis states are

$$|11^{(0)}\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \quad (40)$$

$$|12^{(0)}\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \quad (41)$$

$$|21^{(0)}\rangle = \frac{1}{\sqrt{2}} (|3\rangle - |4\rangle) \quad (42)$$

$$|22^{(0)}\rangle = \frac{1}{\sqrt{2}} (|3\rangle + |4\rangle) \quad (43)$$

with first-order energies given by

$$v_{11} = -2\hbar g \quad (44)$$

$$v_{12} = 2\hbar g \quad (45)$$

$$v_{21} = -2\hbar g \quad (46)$$

$$v_{22} = 2\hbar g \quad (47)$$

In this basis we have

$$V = 2\hbar g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad (48)$$

From this, we see that states $|11^{(0)}\rangle$ and $|21^{(0)}\rangle$ are not coupled to any other states by V . Thus, beyond the first-order energy-shift, they are not perturbed by V .

Thus the problem maps onto a single non-degenerate problem in the $\{|12^{(0)}\rangle, |22^{(0)}\rangle\}$ Hilbert space, for which we have

$$H_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad (49)$$

and

$$V = 2\hbar g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (50)$$

The states to second-order are therefore

$$|11\rangle = |11^{(0)}\rangle \quad (51)$$

$$|12\rangle \approx |12^{(0)}\rangle \left(1 - \frac{2g^2}{\omega_0^2}\right) - |22^{(0)}\rangle \frac{2g}{\omega_0} \quad (52)$$

$$|21\rangle = |21^{(0)}\rangle \quad (53)$$

$$|22\rangle \approx |12^{(0)}\rangle \frac{2g}{\omega_0} + |22^{(0)}\rangle \left(1 - \frac{2g^2}{\omega_0^2}\right) \quad (54)$$

The energies to second-order are

$$E_{11} = \hbar\omega_0 - 2\hbar g \quad (55)$$

$$E_{12} \approx \hbar\omega_0 + 2\hbar g - \frac{4\hbar g^2}{\omega_0} \quad (56)$$

$$E_{21} = 2\hbar\omega_0 - 2\hbar g \quad (57)$$

$$E_{22} \approx 2\hbar\omega_0 + 2\hbar g + \frac{4\hbar g^2}{\omega_0} \quad (58)$$

4. Prove that in order to predict the resonance-frequency shifts of a perturbed quantum system to second-order, one needs only to compute the energy levels to second-order, and that this implicitly requires computing the eigenstates only to first-order.

Then show that in order to compute the shift in the expectation-value of an arbitrary operator to second-order, one must compute the eigenstates to second-order.

The resonance frequency shifts are the differences between the energy eigenvalues, which means they can be computed to second-order directly from the second-order energies. The second-order energy corrections are given in terms of the first-order eigenstates via $E_n^{(2)} = -\langle n^{(0)}|V|n^{(1)}\rangle$.

To compute the expectation value of an operator to second-order, we need to evaluate

$$\begin{aligned}\langle A \rangle &= \langle \psi(t) | A | \psi(t) \rangle \\ &= \sum_{mn} c_m^* c_n \langle m | A | n \rangle e^{i(E_m - E_n)t/\hbar}\end{aligned}\tag{59}$$

expanding to 2nd order gives

$$\begin{aligned}\langle A \rangle &= \sum_m n c_m^* c_n \left[\langle m^{(0)} | A | n^{(0)} \rangle + \lambda \left(\langle m^{(1)} | A | n^{(0)} \rangle + \langle m^{(0)} | A | n^{(1)} \rangle \right) \right. \\ &\quad \left. + \lambda^2 \left(\langle m^{(0)} | A | n^{(2)} \rangle + \langle m^{(1)} | A | n^{(1)} \rangle + \langle m^{(2)} | A | n^{(0)} \rangle \right) \right] \\ &\times e^{i \left[E_m^{(0)} - E_n^{(0)} + \lambda \left(E_m^{(1)} - E_n^{(1)} \right) + \lambda^2 \left(E_m^{(2)} - E_n^{(2)} \right) \right] t/\hbar}\end{aligned}\tag{60}$$

which shows that the second-order eigenstates are required.

5. Consider a pair of identical harmonic oscillators, described by the Hamiltonian

$$H_0 = \hbar\omega(A_1^\dagger A_1 + A_2^\dagger A_2 + 1). \quad (61)$$

Determine the energy eigenvalues and degeneracies of the two lowest energy levels of H_0 .

Let the system be perturbed by the operator

$$V = \hbar\lambda(A_1^\dagger A_2^\dagger + A_1^\dagger A_2 + A_2^\dagger A_1 + A_1 A_2). \quad (62)$$

For the states comprising the two lowest bare energy levels, determine the full energy eigenvalues of $H = H_0 + V$ to second-order and the eigenstates to first-order in λ .

The lowest energy level is the ground state, $E_1^{(0)} = \hbar\omega$, with non-degenerate eigenstate

$$|1^{(0)}\rangle = |00\rangle, \quad (63)$$

The first excited energy level is $E_2^{(0)} = 2\hbar\omega$, with $d_2 = 2$. The degenerate subspace is spanned by the states $\{|01\rangle, |10\rangle\}$.

in basis $\{|01\rangle, |10\rangle\}$, we have

$$V_2 = \hbar\lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (64)$$

For the ground-state, we have

$$E_1^{(1)} = 0 \quad (65)$$

and

$$|1^{(1)}\rangle = - \sum_{\substack{n_1, n_2=0 \\ (n_1, n_2) \neq (0,0)}}^{\infty} |n_1 n_2\rangle \frac{\langle n_1 n_2 | V | 00 \rangle}{\hbar\omega(n_1 + n_2)} \quad (66)$$

with

$$\begin{aligned} \langle n_2 n_2 | V | m_1 m_2 \rangle &= \hbar\lambda \left(\sqrt{(m_1+1)(m_2+1)} \delta_{n_1, m_1+1} \delta_{n_2, m_2+1} + \sqrt{m_2(m_1+1)} \delta_{n_1, m_1+1} \delta_{n_2, m_2-1} \right. \\ &\quad \left. + \sqrt{m_1(m_2+1)} \delta_{n_1, m_1+1} \delta_{n_2, m_2-1} + \sqrt{m_1 m_2} \delta_{n_1, m_1-1} \delta_{n_2, m_2-1} \right) \end{aligned} \quad (67)$$

this becomes

$$|1^{(1)}\rangle = -\frac{\lambda}{2\omega} |11\rangle \quad (68)$$

with the second-order energy then given by

$$E_1^{(2)} = -\frac{\lambda}{2\omega} \langle 00 | V | 11 \rangle = -\frac{\hbar\lambda^2}{2\omega} \quad (69)$$

For the $n = 2$ subspace, the 'good basis' is

$$|21^{(0)}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \quad (70)$$

$$|22^{(0)}\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \quad (71)$$

The first-order energies are

$$E_{21}^{(1)} = -\hbar\lambda \quad (72)$$

$$E_{22}^{(1)} = \hbar\lambda \quad (73)$$

The first-order eigenstates are

$$\begin{aligned} |21^{(1)}\rangle &= -|12\rangle \frac{\langle 12|V|21^{(0)}\rangle}{2\hbar\omega} - |21\rangle \frac{\langle 21|V|21^{(0)}\rangle}{2\hbar\omega} \\ &= -\frac{\lambda}{2\omega} (|12\rangle - |21\rangle) \\ |22^{(1)}\rangle &= -\frac{\lambda}{2\omega} (|12\rangle + |21\rangle) \end{aligned} \quad (74)$$

so that the second-order energies are

$$\begin{aligned} E_{21}^{(2)} &= \langle 21^{(0)}|V|21^{(1)}\rangle \\ &= -\frac{\hbar\lambda^2}{2\sqrt{2}\omega} (\langle 01| - \langle 10|)A_1A_2(|12\rangle - |21\rangle) \\ &= -\frac{\hbar\lambda^2}{\omega} \end{aligned} \quad (75)$$

$$\begin{aligned} E_{22}^{(2)} &= \langle 22^{(0)}|V|22^{(1)}\rangle \\ &= -\frac{\hbar\lambda^2}{\omega} \end{aligned} \quad (76)$$

We have then

$$E_1 \approx \hbar\omega - \frac{\hbar\lambda^2}{2\omega} \quad (77)$$

$$E_{21} \approx 2\hbar\omega - \hbar\lambda - \frac{\hbar\lambda^2}{\omega} \quad (78)$$

$$E_{22} \approx 2\hbar\omega + \hbar\lambda - \frac{\hbar\lambda^2}{\omega} \quad (79)$$

and

$$|1\rangle \approx |00\rangle - \frac{\lambda}{2\omega}|11\rangle \quad (80)$$

$$|21\rangle \approx \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) - \frac{\lambda}{2\omega}(|12\rangle - |21\rangle) \quad (81)$$

$$|22\rangle \approx \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) - \frac{\lambda}{2\omega}(|12\rangle + |21\rangle) \quad (82)$$