

Lecture 10: Coordinate and Momentum Representations

- We will start by considering the quantum description of the motion of a particle in one dimension.
- In classical mechanics, the state of the particle is given by its position and momentum coordinates, x and p .
- In quantum mechanics, we will consider position and momentum as observables and therefore represent them by Hermitian operators, X and P , respectively.
- Based on experimental evidence, we have deduced that:

$$[X, P] = i\hbar I$$

Incompatible Observables

- If two operators do not commute, then an eigenstate of one cannot be an eigenstate of the other.

Proof:

- Strategy: assume opposite and show contradiction

- Assume $|a,b\rangle$ such that

$$A|a,b\rangle = a|a,b\rangle$$

$$B|a,b\rangle = b|a,b\rangle$$

- Operate on $|a,b\rangle$ with $M := [A, B]$

$$M|a,b\rangle = AB|a,b\rangle - BA|a,b\rangle$$

$$= bA|a,b\rangle - aB|a,b\rangle$$

$$= (ba - ab)|a,b\rangle \Rightarrow M|a,b\rangle = 0$$

- either $\det(M) = 0$ or $|a,b\rangle = 0$

- if $\det(M) \neq 0$ then $|a,b\rangle$ does not exist

- Then A and B are compatible

- if $[A, B] = 0$, then $M = 0$, $\{|a,b\rangle\}$ do exist

- Then A and B are incompatible

Coordinate Eigenstates

- Clearly X and P are *incompatible*, thus a particle cannot simultaneously have a well-defined position *and* momentum
- Since X is a Hermitian operator, it follows that its eigenstates form a complete set of unit vectors:

- eigenvalue equation:

$$X|x\rangle = x|x\rangle \quad \forall x \in \mathbb{R}$$

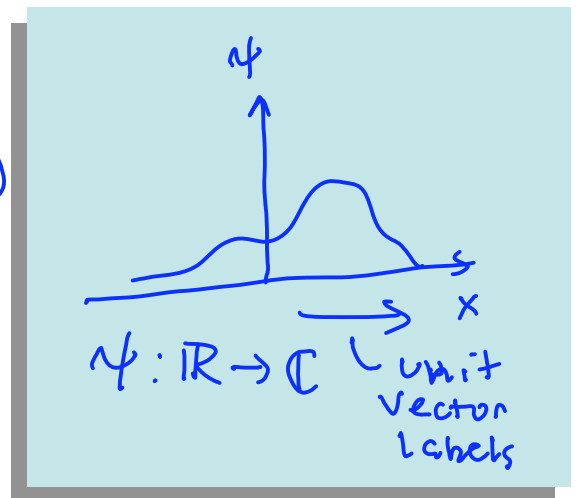
\mathbb{R} = set of real numbers

- orthonormality:

$$\langle x|x'\rangle = \delta(x-x')$$

- closure:

$$\int_{-\infty}^{+\infty} dx |x\rangle\langle x| = I$$



- A state in Hilbert space is completely specified by its components in a 'physical' basis:

$$\text{as } c_n = \langle n|\psi\rangle \Rightarrow \psi(x) = \langle x|\psi\rangle$$

Momentum Representation

- The same logic applies also to momentum eigenstates:

$$P|p\rangle = p|p\rangle \quad \forall p \in \mathbb{R}$$

$$\langle p|p'\rangle = \delta(p-p')$$

$$\int dp |p\rangle \langle p| = I$$

$$\psi(p) := \langle p|\psi\rangle$$

- In order to move fluidly between coordinate and momentum basis, we need to know the transformation coefficients $\langle x|p\rangle$ and $\langle p|x\rangle$:

proof: $\psi(p) = \langle p|\psi\rangle$

$$= \int dx \langle p|x\rangle \langle x|\psi\rangle$$
$$= \int dx \underbrace{\langle p|x\rangle}_{\rightarrow} \psi(x)$$

$$\psi(x) = \langle x|\psi\rangle$$
$$= \int dp \langle x|p\rangle \langle p|\psi\rangle$$
$$= \int dp \underbrace{\langle x|p\rangle}_{\rightarrow} \psi(p)$$

Deriving $\langle x|p\rangle$

- This is the direct derivation
 - Most textbooks use a round-about approach to avoid mathematical subtleties
 - We will just tackle them head on

- start from $P|p\rangle = p|p\rangle$

- hit with $\langle x| \rightarrow \langle x|P|p\rangle = p\langle x|p\rangle$

- insert I: $\int dx' \langle x|P|x'\rangle \langle x'|p\rangle = p\langle x|p\rangle$

- thus we need to know $\langle x|P|x'\rangle$
 - i.e., the matrix elements of P in basis $\{|x\rangle\}$

Derivation of $\langle x|P|x' \rangle$

- All properties of X and P follow
from: $[X, P] = i\hbar I$

- sandwich with $\langle x| \rightarrow$ and $\leftarrow |x' \rangle$

$$\langle x|XP|x' \rangle - \langle x|PX|x' \rangle = i\hbar \langle x|x' \rangle$$

$$(x - x') \langle x|P|x' \rangle = i\hbar \langle x|x' \rangle$$

$$\langle x|P|x' \rangle = i\hbar \frac{\langle x|x' \rangle}{(x - x')}$$

$$\boxed{\langle x|P|x' \rangle = i\hbar \frac{\delta(x - x')}{(x - x')}}$$

- At first glance this looks like a monstrosity, it is zero for $x \neq x'$, but at $x = x'$, it is infinity divided by zero
 - By treating the delta-distribution correctly, we will see that we can easily understand the meaning of this result

Distribution Theory

- Q: is $\delta(x)$ a function?

- A: no, technically it is a 'distribution'
- A 'function' is a mapping from one space onto another:

$$y = f(x)$$

- A 'distribution' is more general than a function
- A distribution is defined only under integration

$$y = \int dx F(x) f(x)$$

- Here, $F(x)$ is the distribution and $f(x)$ is an ordinary function
- For example, the delta-distribution is defined by:

$$\int dx \delta(x - x_0) f(x) = f(x_0)$$

- A distribution can also be defined as the limit of a sequence of functions. All properties of the distribution are by definition, the limiting properties of the sequence

- Example:

$$\delta(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{\pi\sigma^2}} e^{-x^2/\sigma^2}$$

;

∴ properties of $\delta(x)$ are well defined
- sequence is not unique

$\langle x|P|x' \rangle$ is a Distribution

- Our previous result can be written as:

$$\langle x|P|x' \rangle = i\hbar \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{(x-x')^2}{\sigma^2}}$$

- This is well-behaved for any finite σ
 - This proves that it has a clear meaning.
- Insert result into original equation:

(For $\langle x|p \rangle$)

$$\int dx' i\hbar \frac{\delta(x-x')}{(x-x')} \langle x'|p \rangle = p \langle x|p \rangle$$

- Expand $\langle x_{-}|p \rangle$ around $x_{-}=x$:

$$\langle x'|p \rangle = \langle x|p \rangle + (x'-x) \frac{d}{dx} \langle x|p \rangle + \frac{(x'-x)^2}{2} \frac{d^2}{dx^2} \langle x|p \rangle + \dots$$

Continued

$$\int dx' \frac{\delta(x-x')}{(x-x')} \left[\langle x|p \rangle + (x'-x) \frac{d}{dx} \langle x|p \rangle + \frac{(x'-x)^2}{2} \frac{d^2}{dx^2} \langle x|p \rangle + \dots \right]$$

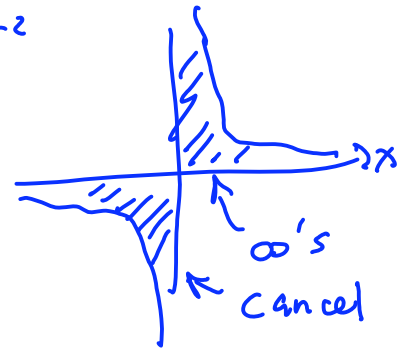
$$= -\frac{i}{\hbar} p \langle x|p \rangle$$

$$\langle x|p \rangle \int dx' \frac{\delta(x-x')}{(x-x')} - \frac{d}{dx} \langle x|p \rangle \int dx' \delta(x-x')$$

$$+ \frac{d^2}{dx^2} \langle x|p \rangle \int dx' (x-x') \delta(x-x') + \dots = \frac{-i}{\hbar} p \langle x|p \rangle$$

$$\int dx' \frac{\delta(x-x')}{(x-x')} = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx' \frac{e^{-\frac{(x-x')^2}{\sigma^2}}}{(x-x')}$$

$$= \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{\pi}\sigma} \int_{-\infty}^{\infty} dx \frac{e^{-x^2/\sigma^2}}{x}$$



$$\therefore \int dx' \frac{\delta(x-x')}{(x-x')} = 0 \quad \text{by parity (odd function)}$$
