

Lecture 11:
X and P, part II

PHY851/fall 2009

Quick review:

$$P|p\rangle := p|p\rangle$$

$$\int dx' \langle x|P|x'\rangle \langle x'|p\rangle = p \langle x|p\rangle$$

$$\langle x|P|x'\rangle = i\hbar \frac{\delta(x-x')}{(x-x')}$$

$$i\hbar \int dx' \frac{\delta(x-x')}{(x-x')} \left[\overset{0}{\langle x|p\rangle} + (x'-x) \frac{d}{dx} \langle x|p\rangle + \frac{(x'-x)^2}{2} \frac{d^2}{dx^2} \langle x|p\rangle + \dots \right]$$
$$= p \langle x|p\rangle$$

$$\frac{d}{dx} \langle x|p\rangle \int dx' \delta(x-x') = \frac{i}{\hbar} p \langle x|p\rangle$$

$$\frac{d}{dx} \langle x|p\rangle = \frac{i}{\hbar} p \langle x|p\rangle$$

$$\int \frac{1}{\langle x|p\rangle} d\langle x|p\rangle = \int \frac{i}{\hbar} p dx$$
$$\ln \langle x|p\rangle + \ln C = \frac{i}{\hbar} px$$

$$\langle x|p\rangle = C e^{\frac{ipx}{\hbar}}$$

Normalization

- The constant C is found from normalization:

$$\text{want: } \langle x|x' \rangle = \delta(x-x')$$

$$\int dp \langle x|p \rangle \langle p|x' \rangle = \delta(x-x')$$

$$\int dp |C|^2 e^{ip(x-x')/\hbar} = \delta(x-x')$$

$$|C|^2 2\pi \delta\left(\frac{x-x'}{\hbar}\right) = \delta(x-x')$$

$$\text{rule: } \delta(ax) = \frac{1}{|a|} \delta(x)$$

$$|C|^2 2\pi \hbar \delta(x-x') = \delta(x-x')$$

- This leads to the results:

$$\langle x|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$
$$\langle p|x \rangle = \langle x|p \rangle^* = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}$$

The momentum operator

- But we have really learned even more.

- By the same logic we must have: *i.e., let $|\rho\rangle \rightarrow |\psi\rangle$*

$$\begin{aligned}\langle x|P|\psi\rangle &= \int dx' \langle x|P|x'\rangle \langle x'|\psi\rangle \\ &= -i\hbar \frac{d}{dx} \langle x|\psi\rangle\end{aligned}$$

$\hookrightarrow = \langle x|\psi\rangle + (x'-x) \frac{d}{dx} \langle x|\psi\rangle + \text{etc...}$

- So when we say:

$$P = -i\hbar \frac{d}{dx}$$

While you can argue that its meaning might be understandable, my point is just that it is not proper use of Dirac notation.

- We really mean:

$$\langle x|P|\psi\rangle = -i\hbar \frac{d}{dx} \langle x|\psi\rangle$$

- In fact, this is all you need to remember

*$|\psi\rangle \Rightarrow |x'\rangle$
 $\hookrightarrow \langle x|P|x'\rangle = -i\hbar \frac{d}{dx} \delta(x-x')$*

equivalent to $i\hbar \frac{\delta(x-x')}{(x-x')}$

I think that a correct statement would be:

$$P = -i\hbar \int dx |x\rangle \frac{d}{dx} \langle x|$$

This is now clearly an operator in Hilbert space, whose meaning is precisely defined by the definition of the derivative.

It then seems reasonable to represent the r.h.s. as:

$$P = -i\hbar \frac{d}{dX}$$

In analogy to:

$$V(X) = -i\hbar \int dx |x\rangle V(x) \langle x|$$

'Coordinate Representation'

- If we make a decision to work in x-basis only, we can abandon Dirac notation, and resort to another common notation:
- The '**coordinate representation**' of the QM theory for a single particle in 1D is related to the Dirac-notation representation via:

$$X \rightarrow x$$

$$P \rightarrow -i\hbar \frac{d}{dx}$$

$$|\psi\rangle \rightarrow \psi(x)$$

$$\langle\psi| \rightarrow \psi^*(x)$$

$$\langle A(X, P) \rangle \rightarrow \int_{-\infty}^{+\infty} dx \psi^*(x) A\left(x, -i\hbar \frac{d}{dx}\right) \psi(x)$$

Gives correct QM description of motion
of a particle

Deriving the wave-equation of a particle

- Start from Schrödinger's Equation:

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$

- For a particle we have:

$$H = \frac{p^2}{2M} + V(x) \quad \leftarrow \text{Empirical result}$$

- Hit with $\langle x|$ from left:

$$i\hbar \frac{d}{dt} \langle x|\psi\rangle = \frac{\langle x|p^2|\psi\rangle}{2M} + \langle x|V(x)|\psi\rangle$$

- Use the properties:

$$\text{from } \langle x|p|\psi\rangle = -i\hbar \frac{d}{dx} \langle x|\psi\rangle$$

$$\langle x|V(x) = \langle x|V(x)$$

- To arrive at:

$$i\hbar \frac{d}{dt} \psi(x) = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} \psi(x) + V(x) \psi(x)$$

- Where, as usual:

$$\psi(x) = \langle x|\psi\rangle$$

Time evolution of the wavefunction

- To find the time evolution of a wavefunction, we can start from the form of the propagator in energy-eigenstate basis:

$$|\Psi(t)\rangle = U(t, t_0) |\Psi(t_0)\rangle$$

$$U(t, t_0) = \sum_n |w_n\rangle e^{-i\omega_n(t-t_0)} \langle w_n|$$

$$|\Psi(t)\rangle = \sum_n |w_n\rangle e^{-i\omega_n(t-t_0)} \langle w_n | \Psi(t_0) \rangle$$

what if all we know is $\Psi(x, t_0)$?

$$\langle x | \Psi(t) \rangle = \sum_n \langle x | w_n \rangle e^{-i\omega_n(t-t_0)} \int dx' \langle w_n | x' \rangle \langle x' | \Psi(t_0) \rangle$$

$$\langle x | \Psi(t) \rangle = \Psi(x, t)$$

let $\langle x | w_n \rangle = \phi_n(x)$ "mode function"

$$c_n(t_0) = \int dx \phi_n^*(x) \Psi(x, t_0)$$

$$\Psi(x, t) = \sum_n \phi_n(x) e^{-i\omega_n(t-t_0)} c_n(t_0)$$

let $\Psi(x, t) := \sum_n \phi_n(x) c_n(t)$

$$\left(c_n(t) = e^{-i\omega_n(t-t_0)} c_n(t_0) \right) \rightarrow \frac{d}{dt} |c_n|^2 = 0$$

Example: a Free Particle

- Consider a free particle of mass M .
- The Hamiltonian is thus:

$$H = \frac{p^2}{2M}$$

- Since $H=H(P)$, it is clear that the momentum eigenstates are also eigenstates of H :

$$\begin{aligned} H |p\rangle &= \frac{1}{2M} p^2 |p\rangle \\ &= \frac{p^2}{2M} |p\rangle \end{aligned}$$

$$E(p) = \frac{p^2}{2M}$$

- Assume that the initial wavefunction, $\psi(x,0)$ is known
- What is $\psi(x,t)$, the wavefunction at a later time?

$$\psi(x,t) = \int dp \int dx' e^{ip(x-x')/\hbar - i\frac{p^2}{2m\hbar}t} \psi(x',0)$$