

Lecture 1: Demystifying 'ħ' and 'i'

• We are often told that the presence of \hbar distinguishes quantum from classical theories.

• One of the striking features of Schrödinger's equation is the fact that the variable, Ψ , is complex, whereas classical theories deal with real variables

QM:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x,t)$$

CM:

$$\frac{d}{dt} x(t) = \frac{\partial}{\partial p} H(x,p)$$

$$\frac{d}{dt} p(t) = -\frac{\partial}{\partial x} H(x,p)$$

CE&M:

$$\frac{d}{dt} \vec{E}(\vec{r},t) = \frac{1}{c^2} \vec{\nabla} \times \vec{B}(\vec{r},t)$$

$$\frac{d}{dt} \vec{B}(\vec{r},t) = -\vec{\nabla} \times \vec{E}(\vec{r},t)$$

• Q: Is \hbar necessary at all?

• By changing units we can of course make \hbar disappear from QM

• But if it is truly fundamental, shouldn't this same choice of units make \hbar appear then in CM?

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) \psi$$

$$\text{let } V(x) = V_0 u\left(\frac{x}{x_0}\right)$$

$$\text{let } t = \frac{\hbar}{V_0} \tau \quad x = x_0 \rho$$

$$i\hbar \frac{V_0}{\hbar} \frac{\partial}{\partial \tau} \psi = -\frac{\hbar^2}{2m x_0^2} \frac{\partial^2}{\partial \rho^2} \psi + V_0 u(\rho) \psi$$

$$i \frac{\partial}{\partial \tau} \psi = -\frac{\hbar^2}{2m x_0^2 V_0} \frac{\partial^2}{\partial \rho^2} \psi + u(\rho) \psi$$

$$\text{let } \boxed{m_0 = \frac{\hbar^2}{x_0^2 V_0}} \quad i \frac{\partial}{\partial \tau} \psi(\rho, \tau) = -\frac{1}{2} \left(\frac{m_0}{m} \right) \frac{\partial^2}{\partial \rho^2} \psi(\rho, \tau) + u(\rho) \psi(\rho, \tau)$$

• If system has natural length scale and energy scale, then \hbar is needed to relate them to the natural mass scale.

•What happens to CM in these units?

$$\frac{\partial}{\partial t} x = \frac{p}{m} \quad \frac{\partial}{\partial t} p = -\frac{\partial}{\partial x} V$$

$$\begin{aligned} V &\rightarrow V_0 u & m &\rightarrow m_0 \mu \\ x &\rightarrow x_0 \rho & p &= p_0 \pi & p_0 &= \frac{m_0 x_0 V_0}{h} \\ t &\rightarrow \frac{h}{V_0} \tau \end{aligned}$$

$$\boxed{\frac{\partial}{\partial \tau} \rho = \frac{\pi}{\mu} \quad \frac{\partial}{\partial \tau} \pi = -\frac{\partial u(\rho)}{\partial \rho}}$$

•Same mass scale makes CM dimensionless as well !

•Q: Are Maxwell's Eq's 'classical' or 'quantum'?

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0$$

•Apply De Broglie hypothesis to Einstein's equation:

$$\begin{aligned} E^2 - c^2 p^2 &= m^2 c^4 \\ E &\rightarrow i\hbar \frac{\partial}{\partial t} & p &\rightarrow -i\hbar \frac{\partial}{\partial x} \\ \left[-\frac{\hbar^2}{c^2} \frac{\partial^2}{\partial t^2} + \hbar^2 \nabla^2 \right] &= -m^2 c^2 \\ \left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi &= -\frac{m^2 c^2}{\hbar^2} \psi \\ \text{let } m &= 0 \end{aligned}$$

- The wavefunction of a massless particle obeys the Maxwell wave equation !
- So is E just the photon wavefunction?
- 'Classical' E&M would be 'quantum' if the photon had mass

My opinion:

- Maxwell's equation is just as 'quantum' as Schrödinger's equation
- 'Classical' EM is ray-optics

• Now, let's look at the 'i' issue:

$$i \frac{\partial}{\partial t} \psi = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \psi + U(x) \psi$$

• Separate ψ into real and imaginary parts:

$$\text{let } \psi(x) = u(x) + i v(x)$$

$$i \frac{\partial}{\partial t} u - \frac{\partial}{\partial t} v = -\frac{1}{2m} u'' - \frac{i}{2m} v'' + U u + i V u$$

$$\frac{d}{dt} u(\vec{r}, t) = H(u, v) v(\vec{r}, t)$$

$$\frac{d}{dt} v(\vec{r}, t) = -H(u, v) u(\vec{r}, t)$$

• No more 'i'

• Structure looks familiar:

• Two conjugate variables

• Symmetric equations

CM:

$$\frac{d}{dt} x(t) = \frac{\partial}{\partial p} H(x, p)$$

$$\frac{d}{dt} p(t) = -\frac{\partial}{\partial x} H(x, p)$$

CE&M:

$$\frac{d}{dt} \vec{E}(\vec{r}, t) = \frac{1}{c^2} \vec{\nabla} \times \vec{B}(\vec{r}, t)$$

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• Can we put an 'i' in CM and make it look more like QM?

$$\text{let } z = \frac{1}{\sqrt{2}} (x + ip) \quad \Leftrightarrow \quad x = z + z^*$$

$$z^* = \frac{1}{\sqrt{2}} (x - ip) \quad p = z - z^*$$

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial}{\partial z} + \frac{\partial z^*}{\partial x} \frac{\partial}{\partial z^*} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z^*} \right)$$

$$\frac{\partial}{\partial p} = \frac{\partial z}{\partial p} \frac{\partial}{\partial z} + \frac{\partial z^*}{\partial p} \frac{\partial}{\partial z^*} = -\frac{i}{\sqrt{2}} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z^*} \right)$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial p} H(x, p) \quad \longrightarrow \quad i \frac{d}{dt} z(t) = \frac{\partial}{\partial z^*} H(z, z^*)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} H(x, p)$$

Newton's second Law

in Q.M.

$$H = \psi^*(x) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \right] \psi(x) \quad \leftarrow \begin{array}{l} \text{Energy} \\ \text{density} \end{array}$$

$$i \frac{d}{dt} \psi(\vec{r}, t) = \frac{\partial}{\partial \psi^*(\vec{r}, t)} H(\psi, \psi^*)$$

•So what is going on?

•The point is that QM is the correct theory

•CM and CE&M are just approximations derived from QM

•Thus they get their structures from QM