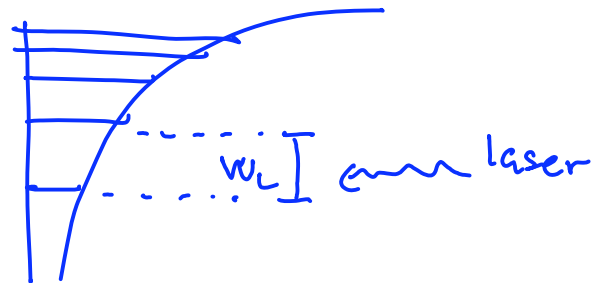


General Study of Two-Level Systems

- Goals:
 - Study ‘Quantum Resonance’ behavior
 - Discuss ‘Avoided Crossings’ and ‘Adiabatic Passage’
 - Study connection between Spin-1/2 and general two-level systems
- Examples of two level systems:
 - A Spin 1/2 particle
 - A ‘two-level atom’



- An atom driven with an oscillating E-field whose frequency closely matches one of the atomic transition frequencies → can ignore other levels
- Particle in a double-well potential
 - E.g. electron in a double quantum dot



- Tunneling couples the lowest level on the left side with the lowest level on the right side

Generic Two-level Hamiltonian

- Consider a system with two quantum energy levels, and a Hamiltonian H_0

- The eigenstates satisfy:

$$H_0|1\rangle = \hbar\omega_1|1\rangle$$

$$H_0|2\rangle = \hbar\omega_2|2\rangle$$

- So that:

$$H_0 = \hbar\omega_1|1\rangle\langle 1| + \hbar\omega_2|2\rangle\langle 2|$$

- In the $\{|1\rangle, |2\rangle\}$ basis, H_0 is represented by the matrix:

$$H_0 = \hbar \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

- The evolution of the system is then:

$$|\psi(t)\rangle = |1\rangle e^{-i\omega_1 t} \langle 1|\psi(0)\rangle + |2\rangle e^{-i\omega_2 t} \langle 2|\psi(0)\rangle$$

Adding a Perturbation

- Suppose we suddenly change the Hamiltonian, e.g. by turning on an external field
 - So that $H \rightarrow H_0 + W$, where

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{pmatrix}$$

- Time Independent Perturbation

$$H = \begin{pmatrix} \hbar\omega_1 + W_{11} & W_{12} \\ W_{12}^* & \hbar\omega_2 + W_{22} \end{pmatrix}$$

- Question#1: What happens to the eigenstates?
 - Question#2: Do we induce transitions from $|1\rangle$ to $|2\rangle$?
- $|2\rangle$

Choosing a Zero of Energy

- Adding a constant times the identity operator to H is equivalent to choosing a new zero of energy
 - Physical predictions unchanged
 - Lets choose $E_0 = -\frac{1}{2}(\hbar\omega_1 + W_{11} + \hbar\omega_2 + W_{22})$

- Which gives

$$H = \begin{pmatrix} \frac{1}{2}(\hbar\omega_1 + W_{11} - \hbar\omega_2 - W_{22}) & W_{12} \\ W_{21} & -\frac{1}{2}(\hbar\omega_1 + W_{11} - \hbar\omega_2 - W_{22}) \end{pmatrix}$$

- Introduce the ‘detuning’ and ‘Rabi frequency’:

$$\Delta = \left(\omega_2 + \frac{W_{22}}{\hbar} - \omega_1 - \frac{W_{11}}{\hbar} \right)$$

$$\Omega = \frac{2W_{21}}{\hbar}$$

- Δ is the energy spacing between the perturbed levels $|\omega_1\rangle$ and $|\omega_2\rangle$
- Ω is the strength of the coupling between them

- The Hamiltonian is then:

$$H = \frac{\hbar}{2} \begin{pmatrix} -\Delta & \Omega^* \\ \Omega & \Delta \end{pmatrix}$$

Most General
Model of a
Two level system
(time - indep.)

- This is the two-level ‘**Rabi Model**’

Physical Examples

- In a spin-1/2 system:

$$\Delta = \frac{\gamma}{\hbar} B_z$$

Zeeman splitting

$$\Omega = \frac{\lambda}{\hbar} B_x$$

Applied B-field along x-axis

- Laser-driven Two level atom:

$$\Delta = (E_u - E_\ell - \hbar\omega_L) / \hbar$$

Difference between laser and transition frequencies

$$\Omega = \frac{2dE_0}{\hbar}$$

Atomic dipole-moment times electric field amplitude

- Double-well potential

$$\Delta = (E_L - E_R) / \hbar$$

Double-well tilt

$$\Omega = \frac{\langle \varphi_L | T | \varphi_R \rangle}{\hbar}$$

Tunneling matrix element

Eigenvalues and Eigenvectors

- Lets Find the Eigenvalues and Eigenvectors:

$$\begin{pmatrix} -\Delta/2 - \omega_{\pm} & \Omega^*/2 \\ \Omega/2 & \Delta/2 - \omega_{\pm} \end{pmatrix} \begin{pmatrix} \langle 1 | \omega_{\pm} \rangle \\ \langle 2 | \omega_{\pm} \rangle \end{pmatrix} = 0$$

$$H|\omega_{\pm}\rangle = \hbar\omega_{\pm}|\omega_{\pm}\rangle \rightarrow (H - \hbar\omega_{\pm}I)|\omega_{\pm}\rangle = 0$$

$$\det \begin{vmatrix} -\Delta/2 - \omega_{\pm} & \Omega^*/2 \\ \Omega/2 & \Delta/2 - \omega_{\pm} \end{vmatrix} = 0$$

$$+ \left(\frac{\Delta}{2} + \omega_{\pm}\right) \left(\omega_{\pm} - \frac{\Delta}{2}\right) - \frac{|\Omega|^2}{4} = 0$$

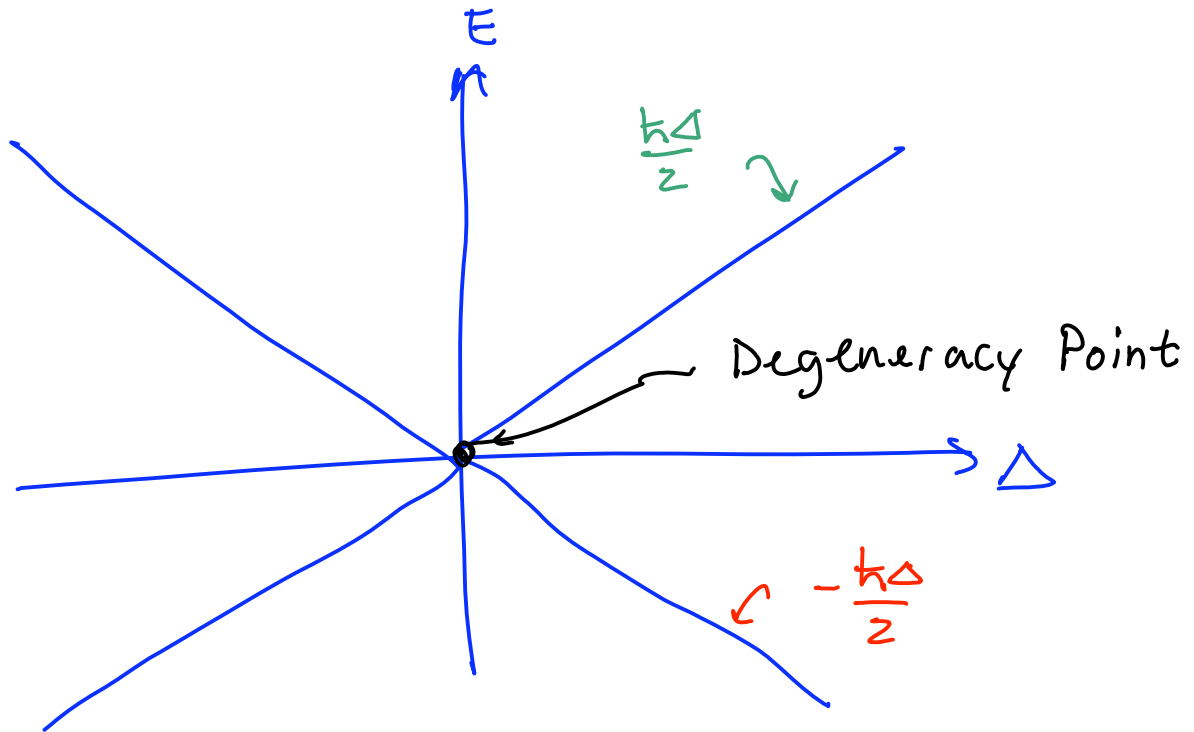
$$\omega_{\pm}^2 - \frac{\Delta^2}{4} - \frac{|\Omega|^2}{4} = 0$$

$$\omega_{\pm} = \pm \frac{1}{2} \sqrt{\Delta^2 + |\Omega|^2}$$

Level Crossing

- Energy spectrum versus Δ for $\Omega=0$:

- no coupling between $|1\rangle$ and $|2\rangle$

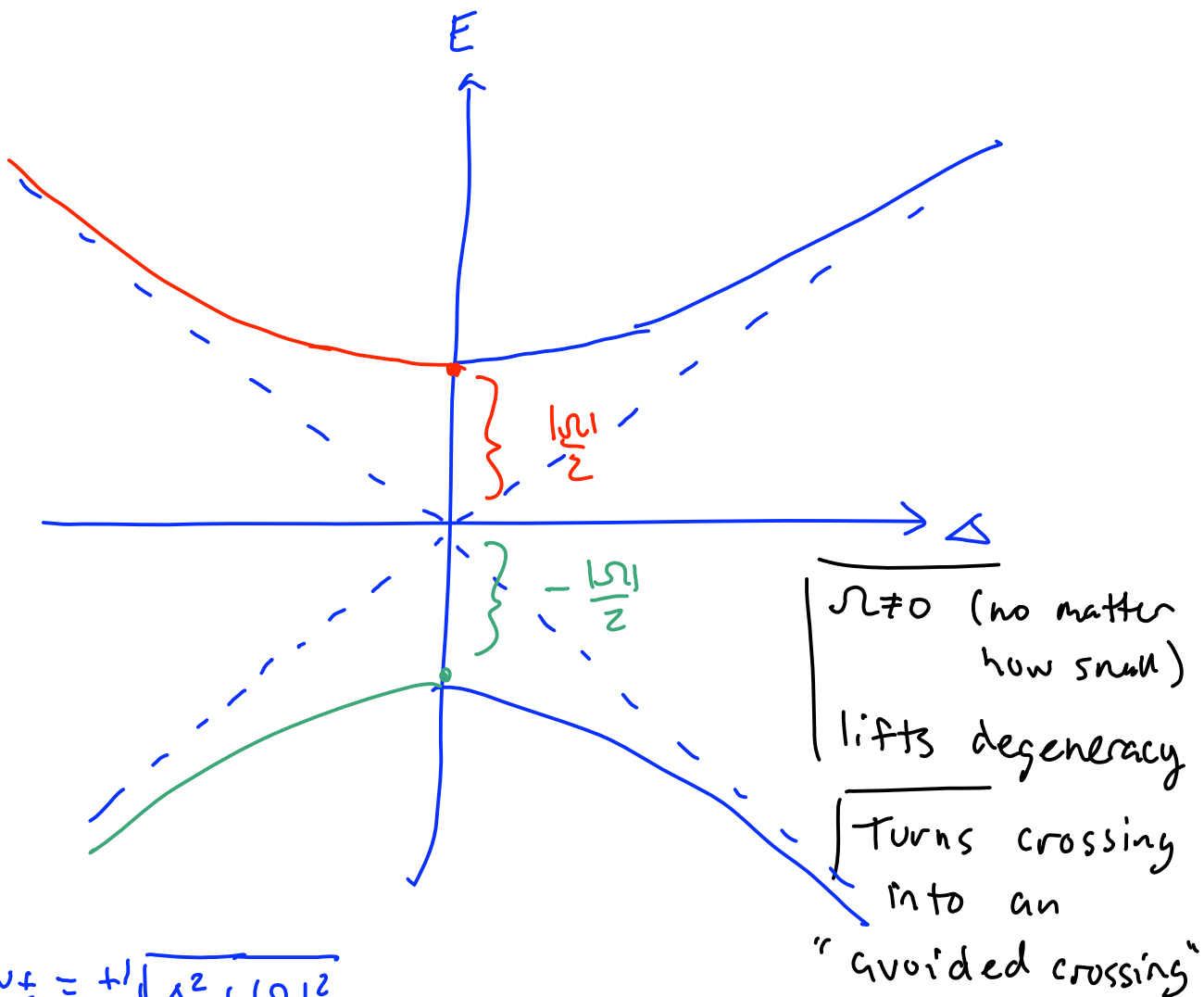


$$H = \frac{\hbar}{2} \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} \quad E_1 = -\frac{\hbar\Delta}{2} \quad \longrightarrow \quad |1\rangle$$
$$E_2 = \frac{\hbar\Delta}{2} \quad \longrightarrow \quad |2\rangle$$

- Degeneracy point \rightarrow Level crossing

Avoided Level Crossing

- Energy spectrum versus Δ for $\Omega \neq 0$:



$$\omega_{\pm} = \frac{\pm 1}{2} \sqrt{\Delta^2 + |\Omega|^2}$$

for $\Delta = 0$

$$\omega_{+} = \frac{|\Omega|}{2}$$

$$\omega_{-} = -\frac{|\Omega|}{2}$$

$$E_{+} = \hbar \omega_{+}$$

— $|w_{+}\rangle$

— $|w_{-}\rangle$

$$E_{-} = \hbar \omega_{-}$$

for $\Delta \rightarrow \pm \infty$

$$\omega_{+} \rightarrow \frac{|\Delta|}{2}$$

$$\omega_{-} \rightarrow -\frac{|\Delta|}{2}$$

$$\hbar \text{gap} = \hbar |\Omega|$$