

## 15. The Simplest Integrals

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### 15.1 INTRODUCTION

In Chapters 5–11 we constructed a Table of Derivatives, which is reproduced in Appendix E. Each entry in the left column is a specific function  $f(x)$ . The corresponding entry in the right column is  $df/dx$ , or  $f'(x)$ .

We can immediately write the indefinite integral for any function in the *right* column of the Table of Derivatives. If we call the function in the right column  $g(x)$  [=  $f'(x)$ ] then the indefinite integral, or antiderivative, of  $g(x)$  is  $f(x) + C$  by the fundamental theorem of calculus:

$$\int g(x)dx = f(x) + C, \quad (15-1)$$

because  $df/dx = g(x)$ . Here  $C$  is an unspecified constant, called the “constant of integration.”

These are the simplest integrals—integrals of functions [ $g(x)$ ] that are easily recognized as derivatives [ $f'(x)$ ].

**Example 1.** What is the integral of  $3x^2$ ?

**Solution.** Look for  $3x^2$  in the right-hand column of the Table of Derivatives in Appendix E. We find  $3x^2$  as the derivative of  $x^3$ . Thus the antiderivative of  $3x^2$  is  $x^3 + C$ :

$$\int 3x^2 dx = x^3 + C, \quad (15-2)$$

because the derivative of the function on the right-hand side of the equation equals the integrand on the left-hand side. The constant  $C$  is arbitrary here, because the derivative of any constant is 0.

**Example 2.** What is the integral of  $\cos x$ ?

**Solution.** Look for  $\cos x$  in the right-hand column of the Table of Derivatives in Appendix E. We find  $\cos x$  as the derivative of  $\sin x$ ; so the antiderivative, or indefinite integral, of  $\cos x$  is  $\sin x + C$ ,

$$\int \cos x dx = \sin x + C. \quad (15-3)$$

Examples 1 and 2 are indefinite integrals. But what if we wanted a *definite* integral? Recall from Sec. 14.5 that we can immediately evaluate a definite integral if we know the indefinite integral. Suppose again that  $g(x) = df/dx$ , so that the indefinite integral of  $g(x)$  is (15-1). Then the definite integral of  $g(x)$  (with arbitrary endpoints  $a$  and  $b$ ) is

$$\int_a^b g(x)dx = f(x)|_a^b = f(b) - f(a) \quad (15-4)$$

by Eq. (14-2). Equation (15-4) introduces a handy notation; for any function  $f(x)$  and domain  $[a, b]$  of  $x$ ,

$$f(x)|_a^b \quad \text{means} \quad f(b) - f(a). \quad (\text{notation})$$

An important point is that the constant of integration  $C$  in (15-1) *drops out* when we calculate the difference of the function at the endpoints,

$$\{f(x) + C\}|_a^b = f(b) + C - [f(a) + C] = f(b) - f(a).$$

**Example 3.** What is the numerical value of the integral of  $e^x$  from  $x = 2$  to  $x = 3$ , accurate to 5 decimal places?

**Solution.** Recall that the derivative of  $e^x$  is  $e^x$ . Therefore the antiderivative of  $e^x$  is  $e^x + C$ .<sup>1</sup> Then by Eq. (15-4) the definite integral is

$$\int_2^3 e^x dx = e^x|_2^3 = e^3 - e^2 = 12.69648. \quad (15-5)$$

**Example 4.**<sup>2</sup> Consider a hyperbola in the first quadrant of the  $xy$  plane, defined by the equation  $xy = 1$ ; a graph of the curve is shown in Fig. 15.1. What is the area of the region bounded above by the hyperbola, bounded below by the  $x$  axis, and bounded on the sides by the lines  $x = 1$  and  $x = 2$ ? The specified region is shaded in Fig. 15.1.

**Solution.** According to the graphical interpretation of the integral, the area  $A$  of the specified region is equal to the integral of  $y(x)$  from  $x = 1$  to  $x = 2$ ,

$$A = \int_1^2 y dx = \int_1^2 \frac{dx}{x}. \quad (15-6)$$

<sup>1</sup>The exponential function is the only function that is its own derivative or its own antiderivative.

<sup>2</sup>This problem was considered previously in Example 3 of Chapter 12.

What is the antiderivative of  $1/x$ ? Recall (or find it in the Table of Derivatives) that the derivative of  $\ln x$  is  $1/x$ ; so the indefinite integral, or antiderivative, of  $1/x$  is

$$\int \frac{dx}{x} = \ln x + C. \quad (15-7)$$

Then the definite integral in (15-6) is

$$\begin{aligned} A &= \ln x \Big|_1^2 = \ln 2 - \ln 1 \\ &= \ln 2 \approx 0.69315. \end{aligned} \quad (15-8)$$

The shaded area under the hyperbola in Fig. 15.1 is  $\ln 2$ .

Generalizing to the domain  $[1, x]$ , the area under the hyperbola from  $x = 1$  to  $x$  is  $\ln x$ . An interesting result is that the *total area* under the hyperbola is infinite. Consider the domain  $[1, x]$ . As  $x$  tends to  $\infty$ , the area,  $\ln x$ , increases to  $\infty$ .

**Example 5.** What is the total area from  $x = 1$  to  $x = \infty$  under the curve in a graph of  $g(x) = 1/x^2$ ? The curve is shown in Fig. 15.2.

**Solution.** The area is

$$\begin{aligned} A &= \int_1^{\infty} g(x) dx = \int_1^{\infty} \frac{dx}{x^2} \\ &= \left. \frac{-1}{x} \right|_1^{\infty} = 0 - (-1) \\ &= 1. \end{aligned} \quad (15-9)$$

Although the region extends to infinity, the total area is finite ( $A = 1$ ). As  $x$  increases, the height of the curve decreases rapidly so that the total area is finite.

## 15.2 HOW TO HANDLE CONSTANT FACTORS

If  $g(x)$  can be found in the right column of the Table of Derivatives in Appendix E, then  $\int g(x)dx = f(x) + C$  where  $f(x)$  is the corresponding function in the left column, because  $f'(x) = g(x)$ . But what about functions that are not identical to any of the forms in the right column of the Table? If a function  $g(x)$  is *similar* to one of the elementary derivatives, but differs by a constant factor, or by a constant multiplying the variable  $x$ , then its integral may be determined using one of the following two theorems.

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**Theorem 15-1.** The integral of  $Kg(x)$  where  $K$  is a constant, is  $K$  times the integral of  $g(x)$ ,

$$\int Kg(x)dx = K \int g(x)dx. \quad (15-10)$$

In other words, if the antiderivative of  $g(x)$  is known, say  $f(x)$ , then the antiderivative of  $Kg(x)$  is also known,  $Kf(x)$ .

**Proof.** We could prove the theorem by writing each integral in (15-10) as the limit of a Riemann sum.<sup>3</sup> But a much simpler proof can be based on the fundamental theorem of calculus. Suppose an antiderivative of  $g(x)$  is  $f(x)$ , so that

$$\int g(x)dx = f(x) + C, \quad (15-11)$$

where  $C$  is a constant of integration. The derivative of the right-hand side of the equation is the integrand  $g(x)$  on the left; that is,

$$\frac{df}{dx} = g(x). \quad (15-12)$$

Now consider the integral of  $Kg(x)$ . Equation (15-10) is true if the derivative of the right-hand side is the integrand  $Kg(x)$  on the left. But the right-hand side of (15-10) is

$$K \int g(x)dx = Kf(x) + C'$$

where  $C'$  is a constant. (Formally,  $C' = KC$ ; but  $C$  is an unspecified constant of integration, so the relation between  $C'$  and  $C$  is irrelevant.) The derivative

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<sup>3</sup>Exercise 5.

of this function is

$$\frac{d[Kf(x)]}{dx} = K \frac{df}{dx} = Kg(x),$$

i.e., the integrand on the left-hand side of (15-10). Hence Theorem 15-1 is true.

Equation (15-10) is often described by this phrase: “The constant can be pulled out of the integral.” We’ll see below how Theorem 15-1 can be used to calculate simple integrals. But first we’ll prove a second useful theorem.

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**Theorem 15-2.** The integral of  $g(\alpha x)$  where  $\alpha$  is a constant, is  $1/\alpha$  times the integral of  $g(\xi)$  where  $\xi = \alpha x$ ,

$$\int g(\alpha x)dx = \frac{1}{\alpha} \int g(\xi)d\xi. \quad (15-13)$$

**Proof.** Again we could prove the theorem by writing each integral as the limit of a Riemann sum,<sup>3</sup> but it is simpler to use the fundamental theorem of calculus. Let  $f(x)+C$  be the integral of  $g(x)$ , as in (15-11). Now consider the integral of  $g(\alpha x)$ . Equation (15-13) is correct if the derivative of the function on the right-hand side of the equation is the integrand on the left-hand side, namely  $g(\alpha x)$ . Now, according to (15-11) the right-hand side of (15-13) is

$$\frac{1}{\alpha} [f(\xi) + C] = \frac{1}{\alpha} f(\alpha x) + C'$$

where we have substituted  $\xi = \alpha x$  and redefined the constant as  $C'$ . The derivative of this function is

$$\begin{aligned} \frac{d}{dx} \left[ \frac{f(\alpha x)}{\alpha} \right] &= \frac{1}{\alpha} \frac{d}{dx} f(\alpha x) \\ &= \frac{1}{\alpha} \left[ \frac{df}{d\xi} \alpha \right] \quad \text{by the chain rule, with } \xi = \alpha x, \\ &= \frac{df}{d\xi} \end{aligned}$$

But we have specified that  $df(x)/dx = g(x)$ , for any value of  $x$ ; therefore  $df/d\xi = g(\xi)$ . Finally, then,

$$\frac{d}{dx} \left[ \frac{f(\alpha x)}{\alpha} \right] = g(\xi) = g(\alpha x),$$

which proves the theorem.

Theorem 15-2 is an example of a *change of variable*. We start on the left-hand side of (15-13) with integration over  $x$ . But the integrand is a function of  $\alpha x$ . So let  $\xi \equiv \alpha x$  be a new variable. The relation of the differentials is

$$d\xi = \alpha dx \quad \text{or} \quad dx = \frac{d\xi}{\alpha}.$$

Therefore, changing the variable of integration to  $\xi$ ,

$$\int g(\alpha x) dx = \int g(\xi) \frac{d\xi}{\alpha} = \frac{1}{\alpha} \int g(\xi) d\xi.$$

In the second equality the constant  $1/\alpha$  has been pulled out of the integral.

As we shall see in the examples, Theorems 15-1 and 15-2 allow us to figure out many simple integrals. If an integrand is *similar* to a function with a known antiderivative, but differs by an overall constant factor  $K$  or by a constant  $\alpha$  multiplying the variable, then the theorems can be applied to perform the integration. After some practice, one learns to do calculations with (15-10) and (15-13) in his head.

### 15.2.1 Examples

**Example 6.** Determine the integral of  $x^2$ .

**Solution.** The function  $x^2$  does not appear in the right column of the Table of Derivatives, but  $3x^2$  appears as the derivative of  $x^3$ . If we multiply the latter function by  $K = 1/3$ , then we have the desired function. By Theorem 15-1,

$$\begin{aligned} \int \frac{1}{3} \times 3x^2 dx &= \frac{1}{3} \int 3x^2 dx = \frac{1}{3} (x^3 + C) \\ &= \frac{1}{3} x^3 + C' \end{aligned} \tag{15-14}$$

where we have used that the integral of  $3x^2$  is  $x^3 + C$ . (The final constant of integration  $C'$  is, like  $C$ , an unspecified constant.) But the *integrand* on the left-hand side of (15-14) is just  $x^2$ , so

$$\int x^2 dx = \frac{1}{3} x^3 + C'. \tag{15-15}$$

After determining an integral we can check the answer. Please verify that the derivative of the right-hand side is equal to the integrand on the left-hand side, as required.

**Generalization.** We can determine the integral of  $x^p$  for any power  $p$  by the same idea. The problem is to calculate  $\int x^p dx$ . Recall that the derivative

of  $x^q$  is  $qx^{q-1}$ . We want the derivative to be  $x^p$ , so set  $q = p + 1$ ,

$$\frac{d}{dx}x^{p+1} = (p+1)x^p. \quad (15-16)$$

This differs from the specified function,  $x^p$ , by the constant factor  $(p+1)$ . So if we multiply  $x^{p+1}$  by  $K_p \equiv 1/(p+1)$  then we have a function whose derivative is  $x^p$ . Thus the general formula for the integral of a power law is

$$\int x^p dx = K_p x^{p+1} + C = \frac{x^{p+1}}{p+1} + C. \quad (15-17)$$

Please show that the result in Example 5 agrees with this general formula.

**A further generalization.** Applying Theorem 15-1 again, for any constant  $A$ ,

$$\begin{aligned} \int Ax^p dx &= A \int x^p dx \\ &= A \left[ \frac{x^{p+1}}{p+1} + C \right] = \frac{Ax^{p+1}}{p+1} + C'. \end{aligned} \quad (15-18)$$

(We could just as well call the final constant of integration  $C$  instead of  $C'$ ; it doesn't matter what we call it because it is an unspecified constant.) Whenever we have calculated an integral we can—and should!—double check that it is correct. It is an easy thing to do. Just verify that the derivative of the integral (the function on the right-hand side of the equation) equals the integrand (the function in the integral on the left-hand side).

**Example 7.** Consider a graph of the function  $\sin x$ . Determine the area bounded above by the curve of  $\sin x$  and below by the  $x$  axis, for  $0 \leq x \leq \pi$ . Figure 15.3 shows a graph of  $\sin x$  with the region under the curve shaded. In other words, the problem is to calculate the area under one hump of a sine curve.

**Solution.** The area under the curve is  $\int_0^\pi \sin x dx$ . What is the antiderivative of  $\sin x$ ? Look for the function  $\sin x$  in the right-hand column of the Table of Derivatives in Appendix E. Actually,  $\sin x$  does not appear in the right-hand column; but  $-\sin x$  does appear as the derivative of  $\cos x$ . Using Theorem 15-1 with  $K = -1$ , we may write

$$\int (-) \sin x dx = - \int \sin x dx. \quad (15-19)$$

Now, the *left-hand* side of (15-19) is  $\cos x + C$  by the fundamental theorem of calculus, because the antiderivative of  $-\sin x$  is  $\cos x$ . Therefore

$$\cos x + C = - \int \sin x \, dx; \quad (15-20)$$

or,

$$\int \sin x \, dx = -\cos x - C. \quad (15-21)$$

(The constant of integration does not need to be designated as  $-C$ ; it could just as well be called  $C'$  or  $C$ , because this unspecified constant will drop out when we calculate a definite integral.) To double check the integration, please verify that the derivative of the integral [the right-hand side of (15-21)] equals the integrand on the left-hand side. The sign is particularly important!

The *definite* integral that we need for this example is

$$\begin{aligned} \int_0^\pi \sin x \, dx &= -\cos x \Big|_0^\pi = -\cos \pi - (-\cos 0) \\ &= 1 + 1 = 2. \end{aligned} \quad (15-22)$$

The final result is that the area under one hump of the sine curve is 2.

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The next two examples make use of Theorem 15-2.

**Example 8.** Find the integral of  $e^{3x}$ .

**Solution.** Recall that the derivative of  $e^\xi$  (with respect to  $\xi$ ) is  $e^\xi$ . We seek now the antiderivative of  $e^{3x}$ , i.e., a function whose derivative is  $e^{3x}$ . The function  $e^{3x}$  doesn't quite work, because its derivative is  $3e^{3x}$  by the chain rule. But  $\frac{1}{3}e^{3x}$  would work. So an antiderivative of  $e^{3x}$  is  $\frac{1}{3}e^{3x}$ , and we may write

$$\int e^{3x} \, dx = \frac{1}{3}e^{3x} + C. \quad (15-23)$$

Please verify that this equation is equivalent to (15-13) in Theorem 15-2, with  $\alpha = 3$  and  $g(\xi) = e^\xi$ .

**Generalization.** What is the integral of  $e^{\alpha x}$  for an arbitrary constant  $\alpha$ ? Applying Theorem 15-2, i.e., a change of variable from  $x$  to  $\xi \equiv \alpha x$ ,

$$\begin{aligned} \int e^{\alpha x} \, dx &= \frac{1}{\alpha} \int e^\xi \, d\xi = \frac{1}{\alpha} (e^\xi + C) \\ &= \frac{1}{\alpha} e^{\alpha x} + C'. \end{aligned} \quad (15-24)$$

Obviously the result (15-23) of Example 8 is just a special case of this general formula!

**Example 9.** Integrate  $\sqrt{1+3.5s}$  for  $s$  from 0 to 7.

**Solution.** We seek to compute  $\int_0^7 \sqrt{1+3.5s} ds$ . The integrand,  $g(s) = \sqrt{1+3.5s}$ , resembles  $\sqrt{1+x}$ , except that the variable  $s$  in  $g(s)$  is multiplied by 3.5. If we can figure out the antiderivative of  $\sqrt{1+x}$  then we can apply Theorem 15-2 to get the integral of  $\sqrt{1+3.5s}$ .

The derivative of  $x^p$  is  $px^{p-1}$ , and the derivative of  $(1+x)^p$  is  $p(1+x)^{p-1}$ . We want the function whose derivative is  $\sqrt{1+x}$ , so we should take  $p = 3/2$ . We must also multiply by  $1/p$  to cancel the constant  $p$  from the derivative. That is,

$$\frac{d}{dx} \frac{2}{3}(1+x)^{3/2} = \sqrt{1+x}. \quad (15-25)$$

Now applying Theorem 15-2, the change of variable from  $s$  to  $x$ ,

$$\begin{aligned} \int \sqrt{1+3.5s} ds &= \frac{1}{3.5} \int \sqrt{1+x} dx \quad (\text{where } x = 3.5s) \\ &= \frac{1}{3.5} \times \frac{2}{3}(1+x)^{3/2} + C \\ &= \frac{2}{10.5} (1+3.5s)^{3/2} + C. \end{aligned} \quad (15-26)$$

Finally, the definite integral specified in the example is

$$\begin{aligned} \int_0^7 \sqrt{1+3.5s} ds &= \frac{2}{10.5} (1+3.5s)^{3/2} \Big|_0^7 \\ &= \frac{2}{10.5} \left\{ (25.5)^{3/2} - 1 \right\} \approx 24.34. \end{aligned} \quad (15-27)$$

**Generalization.** As an exercise<sup>4</sup> the same idea can be used to derive a general formula,

$$\int (1+\alpha s)^p ds = \frac{(1+\alpha s)^{p+1}}{\alpha(p+1)} + C \quad (15-28)$$

where  $\alpha$  and  $p$  are any constants. (As a check, it is easy to show that the derivative of the right-hand side with respect to  $s$  is the integrand on the left-hand side.)

The exercises at the end of the chapter provide additional examples.

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<sup>4</sup>Exercise 6.

## 15.3 A TABLE OF INTEGRALS

It is clear from the previous section that integration of a function  $f(x)$  is straightforward if  $f(x)$  is known as the derivative of another function  $F(x)$ . The fundamental theorem of calculus is

$$\int f(x) dx = F(x) + C, \quad (15-29)$$

$$\text{where } \frac{dF}{dx} = f(x). \quad (15-30)$$

Also, arbitrary constant factors are easily handled. Therefore we can record the simplest integrals in the form of a table. When one of these integrals is needed for a calculation, we can simply look it up in the table. Appendix F provides a short table of indefinite integrals (i.e., antiderivatives). Please verify that the entries in the table follow simply from the basic derivatives in Appendix E, according to (15-29) and (15-30).

In Chapter 17 we'll study *techniques of integration* for calculating integrals that are more complicated than those in Appendix F. Then we will construct a more extensive table of integrals, given in Appendix G.

The ability to use the tables in Appendices F and G, i.e., to identify that an integral takes one of the standard forms, is a valuable skill. But this is not always easy! The tricky complication is that the variables in the desired integral may differ from the variables used in the table.

**Example 10.** Determine the integral of  $\sin(2\pi t/T)$  as a function of  $t$  where  $T$  is a constant.

**Solution.** The table gives

$$\int \sin \alpha x dx = -\frac{1}{\alpha} \cos \alpha x + C. \quad (15-31)$$

To convert this to the example, replace  $x$  by  $t$  and  $\alpha$  by  $2\pi/T$ . The result is

$$\int \sin \left( \frac{2\pi t}{T} \right) dt = -\frac{T}{2\pi} \cos \left( \frac{2\pi t}{T} \right) + C. \quad (15-32)$$

**Example 11.** Use Appendix F (and Theorems 15-1 and 15-2) to integrate  $F(x) \equiv 1/(\alpha^2 x^2 + \beta^2)$  where  $\alpha$  and  $\beta$  are arbitrary constants.

**Solution.** The trick to using a Table of Integrals is to recognize that  $\int F(x)dx$  resembles one of the entries in the table. In Appendix F we find

$$\int \frac{dx}{x^2 + 1} = \arctan x. \quad (15-33)$$

(We'll drop the constant of integration until the end of the calculation.) The difference between  $F(x)$  and  $1/(x^2 + 1)$  is only the presence of the constants  $\alpha$  and  $\beta$ . Can we apply Theorems 15-1 and 15-2? At this point, art and imagination come into the problem! We need to write the integrand in the form

$$\frac{1}{\alpha^2 x^2 + \beta^2} = \frac{C_1}{(C_2 x)^2 + 1}. \quad (C_1 \text{ and } C_2 \text{ are constants.})$$

After staring at this for awhile we see that

$$\frac{1}{\alpha^2 x^2 + \beta^2} = \frac{1}{\beta^2} \frac{1}{(\alpha x/\beta)^2 + 1}.$$

The  $1/\beta^2$  is an overall constant; pull it out of the integral (Theorem 15-1). Then change the variable of integration to  $\xi \equiv \frac{\alpha}{\beta}x$ , replacing  $dx$  by  $\frac{d\xi}{(\alpha/\beta)}$  (Theorem 15-2). So,

$$\int \frac{dx}{\alpha^2 x^2 + \beta^2} = \frac{1}{\beta^2} \frac{1}{\alpha/\beta} \int \frac{d\xi}{\xi^2 + 1}.$$

The integral over  $\xi$  is the table entry (15-33),  $\arctan \xi$ . Finally, then,

$$\int \frac{dx}{\alpha^2 x^2 + \beta^2} = \frac{1}{\alpha\beta} \arctan \frac{\alpha x}{\beta} + C \quad (15-34)$$

where  $C$  is the constant of integration.<sup>5</sup>

### Computer software

Computer programs exist that can determine many integrals analytically. Two commonly used programs are *Mathematica* and *Maple*. The user must input a function into the program, using the format specified by the software. Then the program returns the integral if it has a form that has been programmed into the software. When a computer is available, this software provides a very simple way to evaluate integrals. However, it takes practice to develop skill in using the program. A thorough understanding of integration is necessary in order to use the software effectively. The computer makes mathematics easier, but its use requires a good knowledge of the mathematics.

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<sup>5</sup>Please verify that the answer is correct, i.e., that the derivative of the right-hand side is the integrand on the left.

15.4 EXAMPLES OF INTEGRATION FROM SCIENCE AND  
ENGINEERING

**Example 12. (from kinematics)** A race car has constant acceleration  $a$  on a drag strip. Suppose the final velocity is  $v_f$ , at time  $t_f$ . How far does the car move during the time from  $t = 0$  to  $t = t_f$ ? We will solve this *general* problem formally, using analytic integration.

**Solution.** Acceleration is the rate of change of velocity, so the definition of acceleration is  $a = dv/dt$  where  $v$  is the velocity. By the fundamental theorem of calculus,

$$v(t) = \int_0^t a(t') dt'. \quad (15-35)$$

[Note that the derivative of the integral (the velocity) is equal to the integrand (the acceleration).] We denote the integration variable by  $t'$ , rather than  $t$ , because in (15-35)  $t$  stands for the *endpoint* of the time interval from 0 to  $t$ . In this example  $a$  is constant. We don't yet know its value, but whatever the value of  $a$  the integral is

$$\int_0^t a dt' = at' \Big|_0^t = at - 0 = at. \quad (15-36)$$

The speed is 0 at time 0, and the solution (15-36) has this initial value. At time  $t_f$  the speed is  $v_f$ ; that is,

$$v(t_f) = at_f = v_f, \quad (15-37)$$

which implies

$$a = \frac{v_f}{t_f}. \quad (15-38)$$

We could have obtained this answer more easily by writing  $a = \Delta v / \Delta t = v_f / t_f$ ; however, that would only apply to *constant* acceleration, whereas calculating the integral applies to any time dependence of  $a(t)$ .

Velocity is the rate of change of distance;  $v = dx/dt$  where  $x$  is the *coordinate* of the race car, i.e., the distance from the starting line. Then by the fundamental theorem of calculus,

$$x(t) = \int_0^t v(t') dt'. \quad (15-39)$$

[Again, the derivative  $dx/dt$  of the integral is equal to the integrand  $v(t)$ .] Substituting the velocity function  $v(t') = at'$ ,

$$\begin{aligned} x(t) &= \int_0^t at' dt' = \frac{1}{2} at'^2 \Big|_0^t = \frac{1}{2} at^2 - 0 \\ &= \frac{1}{2} at^2. \end{aligned} \tag{15-40}$$

The distance at time 0 is, of course 0, because the car hasn't started moving yet. The distance traveled at time  $t_f$  is

$$x(t_f) = \frac{1}{2} \frac{v_f}{t_f} t_f^2 = \frac{1}{2} v_f t_f. \tag{15-41}$$

Equation (15-41) is the answer to the original question: the car moves a distance  $v_f t_f / 2$ .

As a numerical example, suppose the race car goes from 0 to 60 mph in 6 s. Then  $v_f = 60 \text{ mi/hr} = 88 \text{ ft/s}$  and  $t_f = 6 \text{ s}$ . The race car moves 264 feet during the 6 seconds of acceleration.

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The next examples were discussed in Chap. 12, as motivation for the importance of the integral. We are now prepared to solve them.

**Example 13. (from astrophysics)** Consider a simple model of the internal structure of a star, in which the density (mass per unit volume) depends linearly on the distance from the center of the star. Figure 15.4 shows the density  $\rho(r)$  as a function of  $r$ . The maximum density  $\rho_0$  occurs at the center of the star,  $r = 0$ . The density decreases monotonically (and, in this simple model, linearly) with distance from the center. At a radius  $R$  the density is 0; this is the radius of the star. Outside radius  $R$  there is no significant amount of matter so  $\rho = 0$  for  $r \geq R$ .

We'll investigate two questions. (a) What is the total mass of the star as a function of  $\rho_0$  and  $R$ ? (b) What does the model imply for the central density of the sun?

**Part (a)** Figure 15.4 shows the star subdivided into thin shells, each with thickness  $\Delta r$ . The number of shells is  $N = R/\Delta r$  because the total radius  $R$  is  $N\Delta r$ . The total mass is the sum of the masses of the thin shells, which we can calculate in the limit  $\Delta r \rightarrow 0$ .

The mass of the shell from radius  $r$  to  $r + \Delta r$  is the density  $\rho$  times the volume. The density varies slightly between  $r$  and  $r + \Delta r$ , but because we'll take the limit  $\Delta r \rightarrow 0$  the variation is negligible. That is, we can just set  $\rho = \rho(r)$  for all points in this thin shell. The approach here is an old idea:

The *variation* in density from  $r$  to  $r + \Delta r$  has a higher order of smallness in the limit  $\Delta r \rightarrow 0$ .

The volume of a spherical shell between radius  $r_1$  and radius  $r_2$  is equal to the volume of the sphere of radius  $r_2$  *minus* the volume of the smaller (inner) sphere of radius  $r_1$ . The volume  $\Delta V$  of the shell from  $r$  to  $r + \Delta r$  is

$$\begin{aligned}\Delta V &= \frac{4}{3}\pi (r + \Delta r)^3 - \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi (r^3 + 3r^2\Delta r + \text{higher order} - r^3) \\ &= 4\pi r^2\Delta r + \text{higher order}\end{aligned}\tag{15-42}$$

where “higher order” indicates terms of order  $(\Delta r)^2$  and  $(\Delta r)^3$ . The higher-order terms are negligible in the limit  $\Delta r \rightarrow 0$ . (The approximation  $\Delta V \approx 4\pi r^2\Delta r$  makes sense: it is the surface area  $4\pi r^2$  times the thickness  $\Delta r$ .) So the total mass is

$$M = \lim_{\Delta r \rightarrow 0} \sum_{k=1}^N \rho(r) 4\pi r^2 \Delta r.\tag{15-43}$$

But this is, by definition, an integral over  $r$ ,

$$M = \int_0^R \rho(r) 4\pi r^2 dr.\tag{15-44}$$

The endpoints are  $r = 0$  (the center) and  $r = R$  (the surface of the star).

The density function in this simple linear model is

$$\rho(r) = \rho_0 \left[1 - \frac{r}{R}\right].\tag{15-45}$$

Figure 15.5 shows a graph of the function. The total mass is

$$\begin{aligned}M &= \int_0^R 4\pi\rho_0 \left[r^2 - \frac{r^3}{R}\right] dr \\ &= 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{r^4}{4R}\right]_0^R \\ &= 4\pi\rho_0 \left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{1}{3}\pi\rho_0 R^3.\end{aligned}\tag{15-46}$$

The integral was evaluated in the second step by the fundamental theorem of calculus. The final result is

$$M = \frac{1}{3}\pi\rho_0 R^3.\tag{15-47}$$

In doing this calculation, the first step is to set up the integral (15-44) and the second step is to compute the integral (15-46). In applications of calculus, the first step is usually more difficult than the second. For instance,

in this example a common *error* would be to say that the mass is  $\int \rho(r)dr$ . Setting up the correct integral (15-44) is crucial.

**Part (b)** The mass and radius of the sun are

$$M = 2 \times 10^{30} \text{ kg} \quad \text{and} \quad R = 7 \times 10^8 \text{ m}. \quad (15-48)$$

The simple linear model implies that the central density  $\rho_0$  is

$$\rho_0 = \frac{3M}{\pi R^3} = \frac{3 \times 2 \times 10^{30} \text{ kg}}{\pi (7 \times 10^8 \text{ m})^3} = 5.6 \times 10^3 \text{ kg/m}^3. \quad (15-49)$$

For comparison, the density of liquid water is  $1 \times 10^3 \text{ kg/m}^3$ .

**Example 14. (from mechanical engineering)** A massive flywheel may be used to store kinetic energy. How much kinetic energy is present in a rotating disk with mass  $M = 450 \text{ kg}$ , radius  $R = 0.6 \text{ m}$ , and height  $h = 0.1 \text{ m}$ , if the period of rotation is  $T = 0.1 \text{ s}$ ?

**Solution.** Figure 15.6 shows the cylindrical flywheel. In the figure, the flywheel is subdivided into  $N$  small cylindrical shells of thickness  $\Delta r$ . The total radius is  $R = N\Delta r$  so the number of shells is  $N = R/\Delta r$ . The total kinetic energy is the sum of the kinetic energies of the elemental shells. To calculate this total we'll calculate the sum in the limit  $\Delta r \rightarrow 0$ .

The kinetic energy  $\Delta K$  of the shell from  $r$  to  $r + \Delta r$  is  $\frac{1}{2}(\Delta m)v^2$  where  $\Delta m$  is the mass of the shell and  $v$  is the speed of points in this shell. The mass is  $\Delta m = \rho\Delta V$  where  $\rho$  is the mass density of the material,

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 h}, \quad (15-50)$$

and  $\Delta V$  is the volume of the shell,

$$\begin{aligned} \Delta V &= \left[ \pi (r + \Delta r)^2 - \pi r^2 \right] h \\ &= 2\pi r h \Delta r + \text{higher order} \end{aligned} \quad (15-51)$$

where "higher order" means terms of order  $(\Delta r)^2$ , which will be negligible in the limit  $\Delta r \rightarrow 0$ . (The approximation  $\Delta V \approx 2\pi r h \Delta r$  makes sense: it is the surface area  $2\pi r h$  of the thin cylindrical shell times the thickness  $\Delta r$ .)

The speed of a point in the flywheel depends on the radius  $r$ . The point at the center has 0 speed. The points on the outer rim have the greatest speed. A point at radius  $r$  travels a distance  $2\pi r$  in each revolution, i.e., in the time  $T$  (= the period of revolution); therefore the speed as a function of  $r$  is

$$v(r) = \frac{2\pi r}{T}. \quad (15-52)$$

The period of revolution is, of course, the same for all points. The speed of a point in the flywheel is proportional to the distance from the center.

Combining these results, the total kinetic energy is

$$\begin{aligned} K &= \lim_{\Delta r \rightarrow 0} \sum_{k=1}^N \frac{1}{2} \rho (2\pi r_k h \Delta r) \left( \frac{2\pi r_k}{T} \right)^2 \\ &= \lim_{\Delta r \rightarrow 0} \sum_{k=1}^N \frac{4\pi^3 \rho h}{T^2} r_k^3 \Delta r. \end{aligned} \quad (15-53)$$

By definition the limit of the sum is an integral,

$$K = \int_0^R \frac{4\pi^3 \rho h}{T^2} r^3 dr. \quad (15-54)$$

This is evaluated by the fundamental theorem of calculus,

$$K = \frac{4\pi^3 \rho h}{T^2} \frac{r^4}{4} \Big|_0^R = \frac{\pi^3 \rho h R^4}{T^2}. \quad (15-55)$$

We may rewrite the result in terms of the mass  $M$  by replacing  $\rho$  by the expression in (15-50). After some algebraic simplification the final result is

$$K = \frac{\pi^2 M R^2}{T^2} \quad (15-56)$$

For the parameters specified in the example, the total kinetic energy is<sup>6</sup>

$$K = \frac{\pi^2 (450 \text{ kg})(0.6 \text{ m})^2}{(0.1 \text{ s})^2} = 1.6 \times 10^5 \text{ J}. \quad (15-57)$$

**Moment of Inertia.** In mechanical engineering the moment of inertia  $I$  of a rotating object is an important parameter. In general, the kinetic energy is  $\frac{1}{2}I\omega^2$  where  $\omega = 2\pi/T$  is the *angular velocity*. Comparing  $\frac{1}{2}I\omega^2$  to the formula (15-56) for a rotating solid cylinder, we see that the moment of inertia of the flywheel is

$$I = \frac{1}{2} M R^2. \quad (15-58)$$

**Example 15. (from electrostatics)** A standard problem in electrostatics is to calculate the electric field and potential due to a given distribution of electric charge. For example, suppose a long wire has charge density  $\lambda = 5 \times 10^{-9} \text{ C/m}$ . What is the strength of the electric field at a distance  $R = 1 \text{ cm}$  from the wire?

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<sup>6</sup>The joule (J) is a unit of energy defined by  $1 \text{ J} = 1 \text{ kg m}^2/\text{s}^2$ .

For simplicity, we approximate the wire as an infinite line. This might seem unrealistic, but in fact it gives a good approximation for points near the wire and far from the ends.

**Solution.** Let the wire be along the  $z$  axis as shown in Fig. 15.7. We have analyzed the problem previously in Example 7 of Chapter 12. The electric field points away from the wire. For a point on the  $x$  axis the field points in the  $x$  direction. The  $x$  component of  $\mathbf{E}$  may be written as an integral over the charged line, as

$$E_x = \int_{-\infty}^{\infty} \frac{(\lambda dz) \sin \theta}{4\pi\epsilon_0 r^2} = \int_{-\infty}^{\infty} \frac{(\lambda dz) R}{4\pi\epsilon_0 (z^2 + R^2)^{3/2}} \quad (15-59)$$

where  $R$  is the perpendicular distance from the wire and we approximate the length as infinite. [See Eq. (12-34) and Fig. 15.7.] According to Theorem 15-1 we can pull any constant factors out of the integral, so

$$E_x = \frac{\lambda R}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + R^2)^{3/2}}. \quad (15-60)$$

The antiderivative of  $(z^2 + R^2)^{-3/2}$  is  $z(z^2 + R^2)^{-1/2} / R^2$ . (Please verify that the derivative of the latter is the former!) Then the definite integral is

$$E_x = \frac{\lambda R}{4\pi\epsilon_0} \left[ \frac{z}{R^2 (z^2 + R^2)^{1/2}} \right] \Bigg|_{-\infty}^{\infty}. \quad (15-61)$$

Evaluating the infinite endpoints is a little tricky; we need the limits

$$\lim_{z \rightarrow \infty} \frac{z}{(z^2 + R^2)^{1/2}} = 1 \quad \text{and} \quad \lim_{z \rightarrow -\infty} \frac{z}{(z^2 + R^2)^{1/2}} = -1.$$

Therefore,

$$\frac{z}{(z^2 + R^2)^{1/2}} \Bigg|_{-\infty}^{\infty} = 1 - (-1) = 2. \quad (15-62)$$

Finally, after simplifying the expression, the electric field is  $E_x \hat{\mathbf{i}}$  where

$$E_x = \frac{\lambda}{2\pi\epsilon_0 R}. \quad (15-63)$$

For the parameters specified in the problem, the electric field strength is

$$E_x = \frac{5 \times 10^{-9} \text{ C/m}}{2\pi \times 8.85 \times 10^{-12} \text{ Cm}^{-1}\text{V}^{-1} \times 0.01 \text{ m}} = 9 \times 10^3 \frac{\text{V}}{\text{m}}. \quad (15-64)$$

For comparison, the dielectric strength of air is  $3 \times 10^6 \text{ V/m}$ . (This is the largest field that can occur in air without dielectric breakdown. For larger

fields the molecules are ionized, the air is a plasma, and current flows.)

Example 14 is rather advanced for an introduction to calculus, because it involves a vectorial quantity—the electric field. However, it is a good example to show how integration is applied to problems in field theory.

**Example 16. (from probability and statistics)** Consider an experiment in which some quantity  $u$  is measured many times. The mean value is  $\bar{u}$  and the standard deviation (= root mean square fluctuation) is  $\sigma$ . What is the probability that a single measurement will lie between  $\bar{u} - \sigma$  and  $\bar{u} + \sigma$ ?

**Solution.** The normal (or, Gaussian) distribution of measurement errors is

$$P(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ - (u - \bar{u})^2 / 2\sigma^2 \right]. \quad (15-65)$$

$P(u)du$  is the probability that a measurement of the quantity would be between  $u$  and  $u + du$ . The expected measurement (= mean of many measurements) is

$$\int_{-\infty}^{\infty} uP(u)du = \bar{u},$$

and the mean square fluctuation is

$$\int_{-\infty}^{\infty} (u - \bar{u})^2 P(u)du = \sigma^2.$$

The probability that a measurement will be between  $\bar{u} - \sigma$  and  $\bar{u} + \sigma$ , i.e., within one standard deviation of the mean, is

$$\begin{aligned} P_1 &= \int_{\bar{u}-\sigma}^{\bar{u}+\sigma} e^{-(u-\bar{u})^2/2\sigma^2} \frac{du}{\sqrt{2\pi\sigma^2}} \\ &= \int_{-\sigma}^{\sigma} e^{-\xi^2/2\sigma^2} \frac{d\xi}{\sqrt{2\pi\sigma^2}}; \end{aligned} \quad (15-66)$$

in the second line the variable has been changed from  $u$  (the measurement) to  $\xi \equiv u - \bar{u}$  (the fluctuation). Using Theorems 15-1 and 15-2 to handle the constants,

$$P_1 = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-x^2/2} dx \quad (15-67)$$

where  $x = \xi/\sigma$ . Finally, the integral can be computed by expanding the exponential in a power series,

$$\int_{-1}^1 e^{-x^2/2} dx = \int_{-1}^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{x^2}{2} \right)^n dx$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \frac{x^{2n+1}}{2n+1} \Big|_{-1}^1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n} \frac{2}{2n+1} \\
&= 2 - \frac{1}{3} + \frac{1}{20} - \frac{1}{168} + \frac{1}{1728} - + \dots \\
&= 1.71. \tag{15-68}
\end{aligned}$$

The probability for  $u$  to lie within  $\pm\sigma$  of  $\bar{u}$  is

$$P_1 = \frac{1.71}{\sqrt{2\pi}} = 0.683. \tag{15-69}$$

In other words, 68% of the measurements would lie within one standard deviation of the mean value.

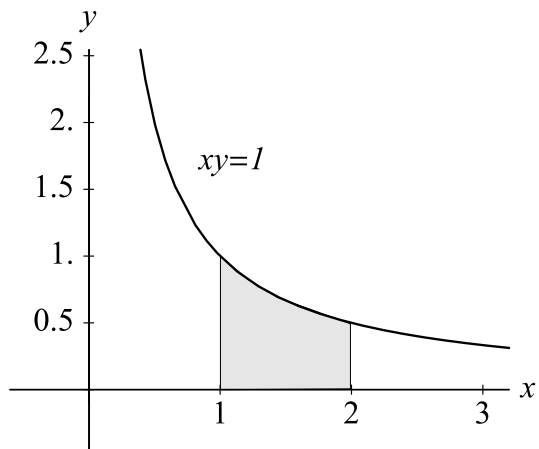


Figure 15.1: Example 4. What is the area of the shaded region under the hyperbola?

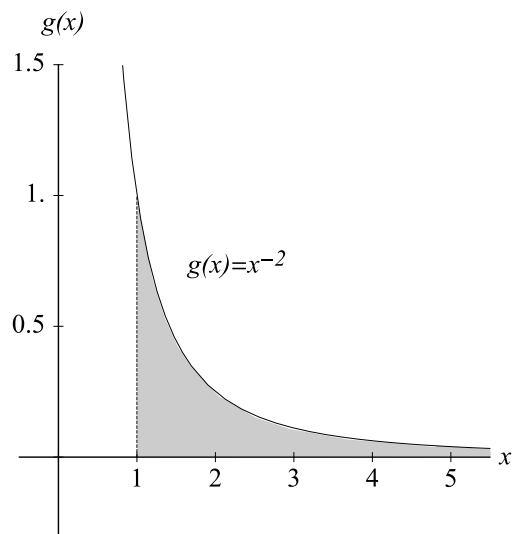


Figure 15.2: Example 5. The shaded area is equal to 1.

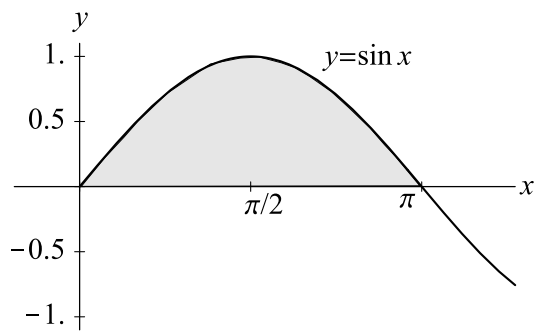


Figure 15.3: Example 7. What is the area of the shaded region—a half-cycle of the sine curve?

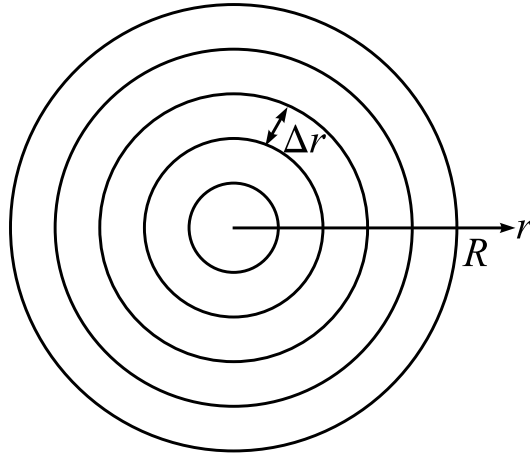


Figure 15.4: Example 12. Imagine a star of radius  $R$  subdivided into many thin spherical shells of thickness  $\Delta r$ .

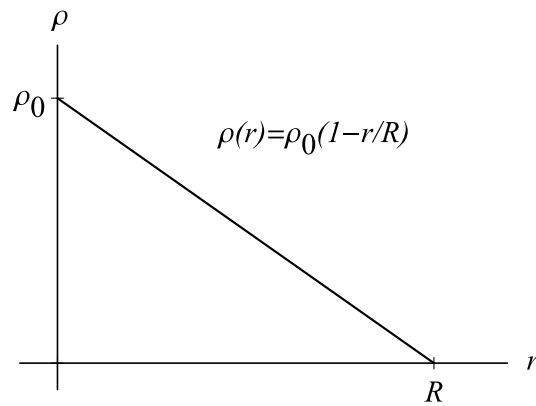


Figure 15.5: Example 12. In this model of the star, the density  $\rho(r)$  decreases linearly from  $\rho_0$  at the center ( $r = 0$ ) to 0 at the surface ( $r = R$ ).

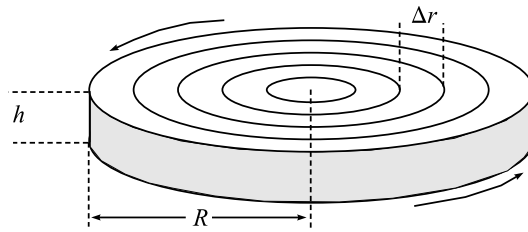


Figure 15.6: Example 13. A rotating flywheel of radius  $R$  and height  $h$ . To calculate the total kinetic energy, the wheel is subdivided into cylindrical shells of thickness  $\Delta r$ . The speed of a point on the wheel is proportional to the distance  $r$  from the center.

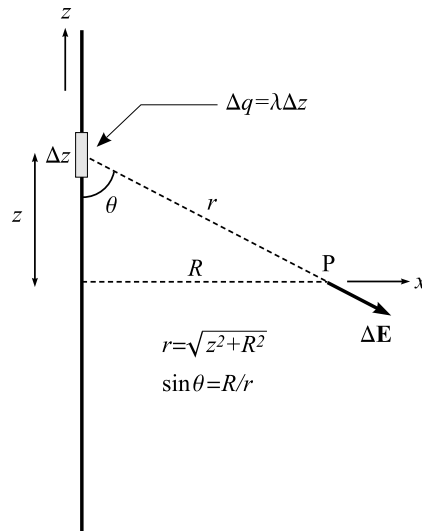


Figure 15.7: Example 14. A charged wire.  $\Delta \mathbf{E}$  is the contribution to the electric field from the small charge element  $\Delta q = \lambda \Delta z$ . The overall field at P is  $\hat{\mathbf{i}} E_x$  in (15-59).

## EXERCISES

**Sec. 1: Introduction**

**15-1.** For each function  $f(x)$  determine the total area under the curve in a graph of  $f(x)$  for the specified domain  $[a, b]$ . Sketch a graph of the function, estimate the area by geometrical methods (try to be accurate!) and compare your estimate to the exact calculated area.

- (a)  $f(x) = e^{-x}$  for  $a = 0$  and  $b = \infty$ .
- (b)  $f(x) = (1 + x)^{-1}$  for  $a = 0$  and  $b = 10$ .
- (c)  $f(x) = 6x$  for  $a = 0$  and  $b = 10$ .
- (d)  $f(x) = 1 + 5x^2$  for  $a = -1$  and  $b = 1$ .
- (e)  $f(x) = (1 + x^2)^{-1}$  for  $a = -\infty$  and  $b = \infty$ . [Hint: look up the derivative of  $\arctan x$ .] Although the graph of  $(1 + x^2)^{-1}$  has no resemblance to a circle, the area under the curve is  $\pi$ .

**15-2.** The equation for a semicircle of radius 1 is  $y = f(x)$  where  $f(x) = \sqrt{1 - x^2}$ .

- (a) Sketch a graph of the function and shade the region bounded by the curve. (What is the area of the shaded region?)
- (b) Prove that the antiderivative of  $f(x)$  is

$$\Phi(x) = \frac{1}{2} [\arcsin x + x\sqrt{1 - x^2}].$$

That is, prove  $\Phi'(x) = f(x)$ .

- (c) Calculate the area of the shaded region of (a) by integration,

$$\text{area} = \int_{-1}^1 f(x) dx.$$

- (d) Explain how these results imply that the area of a circle is  $\pi r^2$ .

**15-3.** Define a function  $J(x)$  by

$$J(x) = \int_0^x k(\sigma) d\sigma.$$

where  $k(\sigma)$  is a given function. Use the definition of the derivative,

$$J'(x) = \lim_{h \rightarrow 0} \frac{J(x+h) - J(x)}{h},$$

to prove that  $J'(x) = k(x)$ .

Hint: In general, for any  $a$ ,  $b$  and  $c$ ,

$$\int_a^b f(\sigma) d\sigma = \int_a^c f(\sigma) d\sigma + \int_c^b f(\sigma) d\sigma;$$

therefore

$$\int_0^{x+h} k(\sigma) d\sigma = \int_0^x k(\sigma) d\sigma + \int_x^{x+h} k(\sigma) d\sigma.$$

**15-4.** The curve in Fig. 15.8 is the hyperbola  $xy = 1$ . Calculate the shaded area.

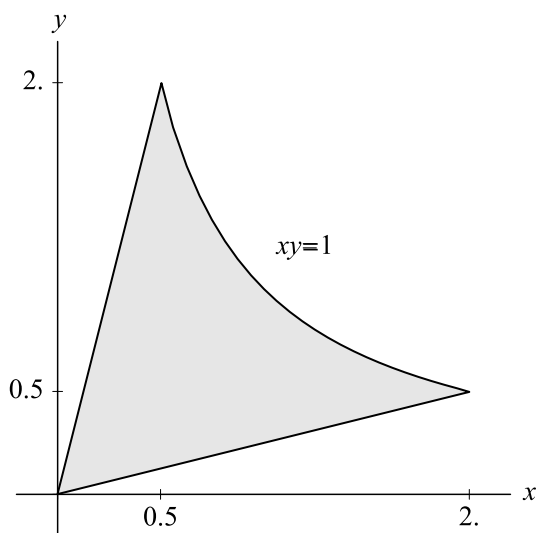


Figure 15.8: Exercise 4.

## Sec. 2: Constant factors

**15-5.** (a) From the basic definition of the integral, as the limit of a Riemann

sum [see Eq. (13-4)], prove Theorem 15-1.

(b) Similarly, prove Theorem 15-2. (Hint: Let  $\xi = \alpha x$ . Draw a picture showing the  $x$  axis and, below it, the  $\xi$  axis. The integration region on the  $x$  axis is some interval  $[a, b]$  which corresponds to an interval  $[\alpha a, \alpha b]$  on the  $\xi$  axis. The subdivision of the  $x$  domain corresponds to a subdivision of the  $\xi$  domain.)

**15-6.** Derive this general integral formula:

$$\int (1 + \alpha s)^p ds = \frac{(1 + \alpha s)^{p+1}}{\alpha(p+1)} + C.$$

**15-7.** Use Theorem 15-2 to prove

$$\frac{1}{a} \int_0^a g\left(\frac{x}{a}\right) dx = \int_0^1 g(x) dx$$

for any function  $g(x)$ .

**15-8.** Evaluate these definite or indefinite integrals using Theorems 15-1 and 15-2 to handle the constant factors. In each case, begin by identifying the integral with one of the standard forms in Appendix F, perhaps modified by constant factors.

(a)  $\int_0^1 \frac{5}{x^2 + 1} dx$

(b)  $\int_0^1 \frac{5}{x^2 + 2} dx$ . Hint: Let  $\xi = \sqrt{2}x$ .

(c)  $\int_0^\infty e^{-u} du$ . Hint: Let  $\xi = -u$ .

(d)  $\int_0^\infty e^{-t/T} dt$ . Hint: Let  $\xi = -t/T$ .

(e)  $\int (ax^2 + bx + c) dx$

(f)  $\int_0^7 (ax^2 + bx + c) dx$

(g)  $\int \frac{dx}{1 - x^2}$

(h)  $\int \frac{dx}{1 - 3x^2}$

### Sec. 3: A Table of Integrals

**15-9.** Look up the antiderivatives for these functions in the Table of Integrals in Appendix F. Where necessary use Theorems 15-1 and 15-2 to handle constant factors. In each case, begin by identifying the integral with one of the standard forms in Appendix F, perhaps modified by constant factors. For extra credit, check the answers using Mathematica or Maple.

(a)  $\int e^{-x} dx$

(b)  $\int e^{-ms} ds$  ( $m$  a constant)

(c)  $\int \cos x dx$

(d)  $\int \cos \omega t dt$

(e)  $\int (3x^2 + 1)^2 dx$

(f)  $\int (x + 2)^7 dx$

(g)  $\int \frac{dx}{x^2 + 1}$

(h)  $\int \frac{du}{7u^2 + 2}$

**15-10.** Use the Table of Integrals in Appendix F to evaluate these definite integrals. Where necessary use Theorems 15-1 and 15-2 to handle constant factors.

(a)  $\int_{-1}^1 \frac{dx}{x^2 + 1}$

(b)  $\int_{-1}^1 \frac{du}{u^2 + 4}$

(c)  $\int_{1/2}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$

(d)  $\frac{1}{a} \int_0^a e^{-x/a} dx$

(e)  $\int_0^\infty x e^{-x^2} dx$

(f)  $\int_0^\infty x e^{-x^2/2} dx$

(g)  $\int_0^{1/4} \sin(2\pi y) dy$

(h)  $\int_0^{\pi/6} \cos \theta d\theta$

(i)  $\int_0^{2\pi} \cos^2 \theta d\theta$ . [Hint:  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ ]

**15-11.** (a) Consider the function  $A(x) = x/\sqrt{1-x^2}$  in the domain  $0 \leq x \leq 1$ . Sketch a graph of the function, shade the region bounded above by the curve and below by the  $x$  axis, and calculate the area.

(b) Consider the function  $B(x) = 1/\sqrt{1-x^2}$  in the domain  $0 \leq x \leq 1$ . Sketch a graph of the function, shade the region bounded above by the curve and below by the  $x$  axis, and calculate the area.

#### Sec. 4: Examples from science and engineering

**15-12.** Suppose the position  $x(t)$  of an object, as a function of time  $t$ , is



$\rho = M/V =$  density, and  $dV =$  volume of the elemental shell.

**15-16.** (a) Derive a formula for the moment of inertia of a sphere (mass  $M$  and radius  $a$ ) about an axis through the center of the sphere. (Hint: Subdivide the sphere into elemental disks as shown in Fig. 15.9. The moment of inertia of an elemental disk is  $dI = \frac{1}{2}(dm)r^2$  where  $r =$  radius,  $dm =$  mass  $= \rho\pi r^2 dz$  and  $\rho =$  density. )

(b) A cylinder and a sphere, of equal radius  $R$  and mass  $M$ , roll down an inclined plane, starting from rest at the same height. Explain why the sphere rolls faster.

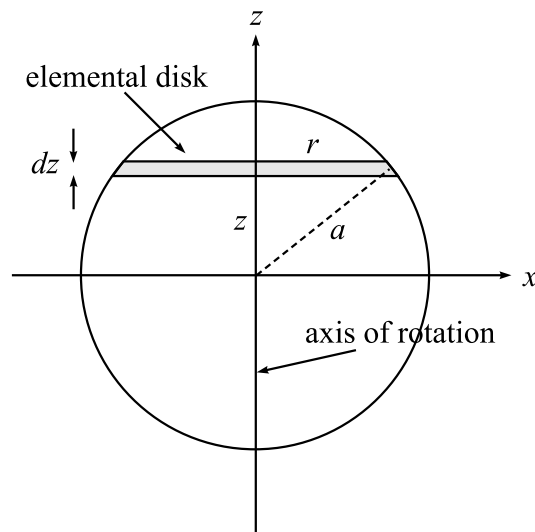


Figure 15.9: Exercise 16. To calculate the moment of inertia of a sphere, subdivide the sphere into elemental disks. The radius of an elemental disk is  $r = \sqrt{a^2 - z^2}$  and the thickness is  $dz$ .

**15-17.** A wire of length  $L$  is uniformly charged with total charge  $Q$  (charge per unit length  $\lambda = Q/L$ ). Determine the electric field on the midplane of the wire as a function of perpendicular distance  $r$  from the wire. (Hint: Let the wire lie on the  $z$  axis from  $-L/2$  to  $L/2$ . Determine the field  $E_x(x)$  for points on the  $x$  axis.) Show that  $E(r)$  behaves as  $r^{-1}$  for  $r \ll L$ , and as  $r^{-2}$  for  $r \gg L$ .

### General Exercises

**15-18.** Consider a set of small masses  $m_i$  occupying points  $(x_i, y_i)$  in the  $xy$  plane. (The index  $i = 1, 2, 3, \dots, N$  labels the masses.) The center of mass point  $(x_c, y_c)$  is the “average point” weighted by the masses,

$$x_c = \frac{\sum_{i=1}^N m_i x_i}{\sum_{i=1}^N m_i} \quad \text{and} \quad y_c = \frac{\sum_{i=1}^N m_i y_i}{\sum_{i=1}^N m_i}.$$

Similarly, for a continuous distribution of mass, i.e., a flat plate in the  $xy$  plane, the coordinates of the center of mass point are

$$x_c = \frac{1}{M} \int x \, dm \quad \text{and} \quad y_c = \frac{1}{M} \int y \, dm$$

where  $dm$  denotes a mass element and  $M = \int dm$  is the total mass. Find the center of mass point of a flat plate in the shape of a right isosceles triangle. (Hint: Set up a coordinate system with the hypotenuse of the triangle on the  $x$  axis. Subdivide the plate into elemental strips parallel to the  $x$  axis.)

**15-19.** *Work and potential energy.* The elastic force on a mass  $m$  is

$$F(x) = -kx \quad (\text{Hooke's law})$$

where  $k$  is the *spring constant*. The mass moves on the  $x$  axis. A positive force is toward the right, i.e., toward increasing  $x$ .

(a) Show that  $x = 0$  is a stable equilibrium point: for either  $x > 0$  or  $x < 0$  the force pulls toward  $x = 0$ .

(b) The work  $W$  done by the force as  $m$  moves from  $x = a$  to  $x = b$  is defined by  $W = \int_a^b F(x) dx$ . Calculate the work done if  $m$  moves from  $x = a$  to  $x = 0$ .

(c) The potential energy  $U(x)$  as a function of the position  $x$  of  $m$  is defined as the work done by the force if  $m$  moves from position  $x$  to the equilibrium position. Determine  $U(x)$ .

(d) Prove that  $F(x) = -U'(x)$ .

**15-20.** A mathematical farmer has decided that his potato patch should be bounded by a square-root curve, because the potato is a root vegetable. His field has dimensions 100 ft  $\times$  100 ft, and the potato patch is shown in Fig. 15.10. What is the area of the potato patch?

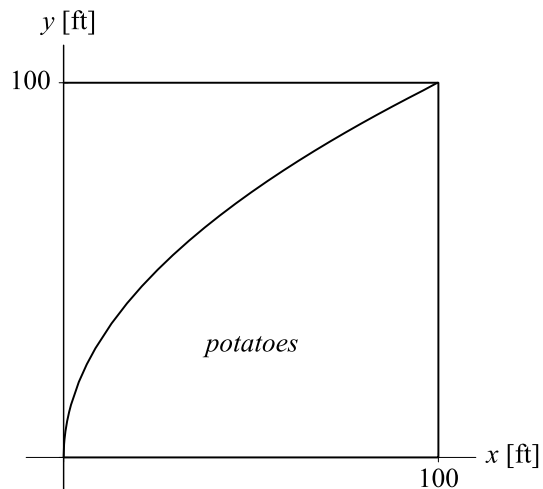


Figure 15.10: Exercise 20. A potato patch bounded by a square root curve,  $y = 10\sqrt{x}$ .

**15-21.** Sketch the unit circle in the  $xy$  plane centered at the origin; the equation for the circle is  $x^2 + y^2 = 1$ . Sketch a vertical line at  $x = 1/2$ . The region inside the circle with  $x > 1/2$  is called a *chord* of the circle. What is the area of this chord? Determine the answer in two ways: (i) by integration, and (ii) by regarding the chord as the circular segment minus a triangle. [Answer:  $\pi/3 - \sqrt{3}/4$ ]

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