The Continuity Equation

An important equation in electrodynamics is the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{1}$$

where $\mathbf{J}(\mathbf{x}, t) =$ current density and $\rho(\mathbf{x}, t) =$ charge density. (The units of $\mathbf{J}$ and ρ are $\mathrm{A/m^2}$ and $\mathrm{C/m^3}$, respectively, so (1) is dimensionally correct.) This equation expresses the local conservation of charge. In integral form it is

$$\oint_S \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt} Q_{\text{enclosed}};$$

the current outward through S must equal the rate of decrease of charge enclosed by S.

Equations similar to (1) appear in many places in theoretical physics. In each case a conservation law is expressed. For example, conservation of mass in fluid mechanics is the equation

$$\nabla \cdot (\rho_m \mathbf{v}) = -\frac{\partial \rho_m}{\partial t}$$

where $\rho_m(\mathbf{x}, t) =$ mass density and $\mathbf{v}(\mathbf{x}, t) =$ fluid velocity. The mass flux is $\rho_m \mathbf{v}$ (= mass per unit area per unit time).

In quantum mechanics, the probability density is $\rho(\mathbf{x}, t) = |\psi(\mathbf{x}, t)|^2$ where $\psi(\mathbf{x}, t)$ is the Schroedinger wave function. Probability is locally conserved, so there is an equation with the same form as (1) expressing conservation of probability. The probability current density is

$$\mathbf{J}(\mathbf{x}, t) = \frac{\hbar}{2im} \left[\psi^* \nabla \psi - (\nabla \psi^*) \psi \right].$$

A nice exercise in quantum mechanics is to verify (1).

The equation for energy conservation in an electromagnetic field is analogous to (1). It is shown in Chapter 11 that

$$\nabla \cdot \mathbf{S} = -\frac{\partial u}{\partial t}; \tag{2}$$

here u is the energy density,

$$u = \frac{1}{2} \rho_m v^2 + \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2,$$

and $\mathbf{S}$ is the energy flux,

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}.$$

Equation (2) is called Poynting's theorem, and $\mathbf{S}(\mathbf{x}, t)$ is the Poynting vector.