3. Basic Principles of Electrostatics
Self-test answers

1. This problem is most easily solved by applying Gauss's theorem. (*Another method that may seem more direct—integrating over the charge density—is much more difficult mathematically.*) Consider as Gaussian surface a sphere S of radius r, and

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = Q_{\text{enclosed}}/\epsilon_0.$$

By spherical symmetry the field is radial, $\mathbf{E}(\mathbf{x}) = E_r(r)\widehat{\mathbf{r}}$, so $\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi r^2 E_r(r)$. The charge enclosed is Q if $r > a$, and Qr^3/a^3 if $r < a$. Thus, because of the spherical symmetry,

$$E_r(r) = \frac{Q_{\text{enclosed}}}{4\pi\epsilon_0 r^2} = \begin{cases} Q/(4\pi\epsilon_0 r^2) & \text{if} \quad r > a \\ Qr/(4\pi\epsilon_0 a^3) & \text{if} \quad r < a. \end{cases} \tag{1}$$

The potential, specified to be 0 at $r = \infty$, is

$$V(r) = \int_r^\infty E_r dr = \begin{cases} Q/(4\pi\epsilon_0 r) & \text{if} \quad r > a \\ Q(3a^2 - r^2)/(8\pi\epsilon_0 a^3) & \text{if} \quad r < a. \end{cases} \tag{2}$$

Please sketch graphs of (1) and (2), to make sure that you understand the results.

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2. Consider the arbitrary point $(0, 0, z)$ with $z > 0$. By symmetry, $\mathbf{E}$ is in the $\widehat{\mathbf{k}}$ direction there. Since this problem does not have complete cylindrical symmetry (it lacks translation invariance in z) we cannot solve the problem by Gauss's law. We must use superposition, i.e., integrate over the charge distribution,

$$\begin{aligned} E_z(0,0,z) &= \int_{-\infty}^0 \frac{\lambda dz'}{4\pi\epsilon_0 (z-z')^2} \\ &= \frac{\lambda}{4\pi\epsilon_0} \left. \frac{1}{z-z'} \right|_{z'=-\infty}^0 = \frac{\lambda}{4\pi\epsilon_0 z}, \end{aligned}$$

where λ is the charge per unit length.

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3. By Poisson's equation the charge density is

$$\rho(\mathbf{x}) = \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{3\epsilon_0 K a^2}{4\pi \left(r^2 + a^2\right)^{5/2}}.$$

Please verify that the total charge is $4\pi\epsilon_0 K$, independent of a. If $a \to 0$, then $\mathbf{E}(\mathbf{x})$ approaches the field of a point charge.

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4. The electric field at an arbitrary point is Eq. (3.100),

$$\mathbf{E}(\mathbf{x}) = \frac{3\widehat{\mathbf{r}}\,(\widehat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}}{4\pi\epsilon_0 r^3}.$$

Thus at the two specified points,

$$\begin{aligned}
\mathbf{E}(d,0,0) &= \frac{-p_0\widehat{\mathbf{k}}}{4\pi\epsilon_0 d^3}, \qquad (\text{note} : \widehat{\mathbf{r}} = \widehat{\mathbf{i}}) \\
\mathbf{E}(0,0,d) &= \frac{2p_0\widehat{\mathbf{k}}}{4\pi\epsilon_0 d^3}. \qquad (\text{note} : \widehat{\mathbf{r}} = \widehat{\mathbf{k}})
\end{aligned}$$

The fields at $(d,0,0)$ and $(0,0,d)$ point in opposite directions, with strengths in the ratio 1:2. (Please verify the directions by inspecting Figure 3.14.)

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5. The energy levels of the hydrogen atom may be calculated from the Schroedinger equation with potential energy $V(r) = -e^2/(4\pi\epsilon_0 r)$, which corresponds to the Coulomb force. The cross section for Rutherford scattering may be calculated from the Coulomb force between the α particle and the atomic nucleus.

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6. (a) Solve this problem by superposition of: (1) the field $\mathbf{E}_1$ that would be produced by a surface charge density $+\sigma$ on the entire xy plane, and (2) the field $\mathbf{E}_2$ of a disk with radius R, surface charge density $-\sigma$, and centered at the origin.

Use Eq. (3.46) for $\mathbf{E}_1$. For $\mathbf{E}_2$ use the result of Example 9 in Chapter 3 (but making the changes $\sigma \to -\sigma$ and $a \to R$ for this exercise). The result is

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 = \frac{\sigma}{2\epsilon_0}\widehat{\mathbf{k}} - \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\widehat{\mathbf{k}} \\
&= \frac{\sigma z}{2\epsilon_0\sqrt{R^2+z^2}}\widehat{\mathbf{k}}.
\end{aligned}$$

Note that $\mathbf{E}$ is zero at the origin (center of the hole) and the field strength increases monotonically with z.

(b) Expanding the result of (a) for $z \gg R$ gives

$$E_z(0,0,z) = \frac{\sigma}{2\epsilon_0}\left(1 + \frac{R^2}{z^2}\right)^{-1/2} \sim \frac{\sigma}{2\epsilon_0} - \frac{\sigma R^2}{4\epsilon_0 z^2}\cdots.$$

The first term in the expansion is the field of a uniformly charged plane, and the second term is the field of a point charge $-\pi R^2\sigma$ at the origin, as we expect.

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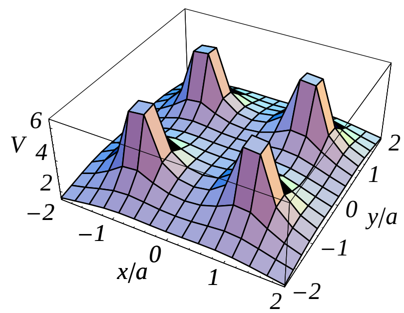
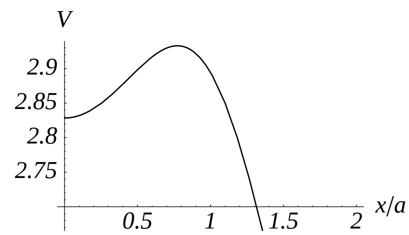
7. The figure on the left shows a 3D surface plot of $V(x, y)$.





The work done in moving a charge q from $(0, 0)$ to $(a, 0)$ (say at constant velocity) is $W = -q \int \mathbf{E} \cdot d\mathbf{x}$. To evaluate the line integral would be quite hard. Much easier is to use the potential. The work is $W = q\Delta V$, i.e.,

$$\begin{aligned} W &= q\,[V(a,0) - V(0,0)] \\ &= \frac{qQ}{4\pi\epsilon_0 a}\left[\frac{2}{1} + \frac{2}{\sqrt{5}} - \frac{4}{\sqrt{2}}\right] = 0.066\,\frac{qQ}{4\pi\epsilon_0 a}. \end{aligned}$$

The graph on the right shows $V(x, 0, 0)$ versus x. There is an energy barrier. This does not violate Earnshaw's theorem (see Exercise 5.26) because we have only considered motion in the xy plane. There is no energy barrier for motion along the z axis.