## Fourier Series

If $f(x)$ is a periodic function with period $c$, i.e., $f(x+c) = f(x)$, then $f(x)$ can be expanded in sinusoidal functions as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{2\pi n x}{c} + b_n \sin \frac{2\pi n x}{c} \right]. \tag{1}$$

Because $f(x)$ is periodic, any real interval of length $c$ may be taken as the primary domain of $f(x)$; the two intervals most commonly encountered are $(0, c)$ and $(-c/2, c/2)$. Using $(-c/2, c/2)$ as the primary domain, the Fourier coefficients are

$$\begin{aligned} a_n &= \frac{2}{c} \int_{-c/2}^{c/2} f(x) \cos \frac{2\pi n x}{c} dx, \\ b_n &= \frac{2}{c} \int_{-c/2}^{c/2} f(x) \sin \frac{2\pi n x}{c} dx. \end{aligned}$$

Fourier series appear in the solution of boundary-value problems with rectangular boundaries, such as Examples 1 and 2 in Chapter 5. In such a problem $f(x)$ is not periodic, but restricted to a bounded domain. However, the Fourier series applies because the domain of the function may be artificially extended to all $x$ by making the function periodic beyond the domain of the boundary-value problem. For example, in Example 2, the physical domain is $0 \le z \le \ell_3$; the solution is expressed as a Fourier series for an odd function $[f(-z) = -f(z)]$ with period $c = 2\ell_3$.

When a Fourier series occurs in a boundary-value problem, the symmetry of the problem may imply that some coefficients, $a_n$ or $b_n$, must be 0. Applying the symmetry simplifies the solution of the problem. For example, if $f(x)$ is known to be even in $x$, then $b_n = 0$.

Another form in which a Fourier series might appear is for a function defined on the domain $(-d, d)$. In this case the period is $2d$, and so we replace $c$ in (1) by $2d$; the expansion is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{\pi n x}{d} + b_n \sin \frac{\pi n x}{d} \right]. \tag{2}$$

For example, a square wave $S(x)$ equal to $-1$ for $-d \le x \le 0$ and $+1$ for $0 \le x \le d$, is expanded as a Fourier series as

$$S(x) = \frac{4}{\pi} \sum_{n,\text{odd}} \frac{1}{n} \sin \frac{n\pi x}{d}.$$

By definition $S(d/2) = 1$. Substituting this value into the series we find the beautiful formula

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$