Electromagnetism
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5. General Methods for Laplace's Equation
Self-test answers

1. Using separation of variables and the boundary conditions, the potential must have the form

$$V(\mathbf{x}) = \sum_{m,n} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \frac{\cosh \gamma_{mn} z}{\cosh \gamma_{mn} L/2}$$

where $\gamma_{mn}^2 = (m\pi/a)^2 + (n\pi/b)^2$. The end conditions require

$$V_0 = \sum_{m,n} c_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

which must be used to determine the constant coefficients c_{mn}. The relevant orthogonality conditions involve integrals with $0 \le x \le a$ and $0 \le y \le b$. Then, projecting out c_{mn},

$$\begin{aligned} c_{mn} &= \frac{4}{ab} \int_0^a \int_0^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) V_0 dx dy \\ &= \begin{cases} \dfrac{16 V_0}{\pi^2 ab} \dfrac{1}{mn} & \text{if } m \text{ and } n \text{ are odd,} \\ 0 \ \ \text{otherwise.} \end{cases} \end{aligned}$$

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2. For $r \ge a$ the solution of Laplace's equation is

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos\theta).$$

The term with $\ell = 1$ is

$$A_1 r \cos\theta + B_1 \frac{\cos\theta}{r^2}.$$

We must set $A_1 = -E_0$ to get the potential $-E_0 z$ of the applied field. Then the boundary condition $V(a, \theta) = 0$ implies $B_1 = E_0 a^3$. No other ℓ values are needed to satisfy the boundary conditions, so by the uniqueness theorem the potential is

$$V(r, \theta) = -E_0 r \cos\theta + \frac{E_0 a^3}{r^2} \cos\theta.$$

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3. (a) $V(x, y) = x^2 - y^2 + \text{constant}$. The equipotentials are hyperbolas with asymptotes $y = -x$ and $y = x$. The boundary-value problem corresponding to this potential is charged conducting half planes on the orthogonal surfaces $y = -x$ and $y = x$.

(b) $V(x, y) = 2xy + \text{constant}$. The equipotentials are hyperbolas with asymptotes $y = 0$ and $x = 0$. The boundary value problem corresponding to this potential is charged conducting half planes on the positive x and y axes.