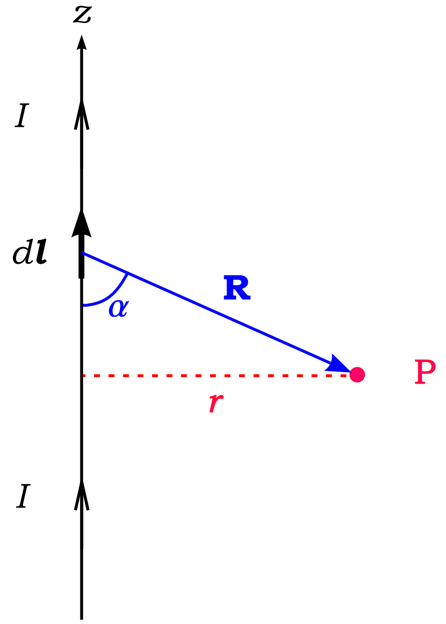
Magnetic Field of a Long Straight Wire

In 1819, H. C. Oersted observed that a current-carrying wire causes deflection of a compass needle. What is the magnetic field? The most basic example is a current I in a long straight wire. To calculate $\mathbf{B}$ for points near the wire we may neglect other currents in the circuit and approximate the length of the wire as infinite.



The figure shows the current I along the z axis, and a point P where we will determine $\mathbf{B}$. According to the Biot-Savart law, the contribution to $\mathbf{B}$ from the segment of wire $d\boldsymbol{\ell}$, located at z, is

$$d\mathbf{B} = \frac{\mu_0 I d\boldsymbol{\ell} \times \widehat{\mathbf{R}}}{4\pi R^2}$$

where $\mathbf{R}$ is the vector from $d\boldsymbol{\ell}$ to P, as shown.

Note that $R = \sqrt{r^2 + z^2}$, and

$$d\boldsymbol{\ell} \times \widehat{\mathbf{R}} = dz\, \sin\alpha\, \widehat{\boldsymbol{\phi}} = dz\, \frac{r}{R}\, \widehat{\boldsymbol{\phi}}.$$

The direction of $d\boldsymbol{\ell} \times \widehat{\mathbf{R}}$ is into the page at P, i.e., azimuthal; the field "curls around" the current. The full field is $\int d\mathbf{B}$,

$$\mathbf{B} = \frac{\mu_0 I r}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(r^2 + z^2)^{3/2}}\, \widehat{\boldsymbol{\phi}}.$$

The integral is elementary, but requires some care with the infinite end points:

$$\text{integral} = \left. \frac{z}{r^2\,(r^2 + z^2)^{1/2}} \right|_{z=-\infty}^{z=\infty} = \frac{2}{r^2}.$$

Thus,

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\, \widehat{\boldsymbol{\phi}}.$$

Note that for a circle C,

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} \frac{\mu_0 I}{2\pi r} r\, d\phi = \mu_0 I,$$

an example of Ampère's law.