Uncertainties of Parton Distribution Functions

Dan Stump
Michigan State University

Outline

1/ Introduction
2/ Compatibility of Data
3/ Uncertainty Analysis
4/ Examples of PDF uncertainty
5/ Outlook

The systematic study of uncertainties of PDF's developed slowly. Pioneers…

J. Collins and D. Soper, CTEQ Note 94/01, hep-ph/9411214.

C. Pascaud and F. Zomer, LAL-95-05.

M. Botje, Eur. Phys. J. C 14, 285 (2000).

Today many groups and individuals are involved in this research.

Current research on PDF uncertainties

CTEQ group
at Michigan State (J. Pumplin, D. Stump, WK. Tung,
HL. Lai, P. Nadolsky, J. Huston, R. Brock)
and others (J. Collins, S. Kuhlmann, F. Olness, J. Owens)

MRST group (A. Martin, R. Roberts, J. Stirling, R. Thorne)

Fermilab group (W. Giele, S. Keller, D. Kosower)

S. I. Alekhin

V. Barone, C. Pascaud, F. Zomer; add B. Portheault

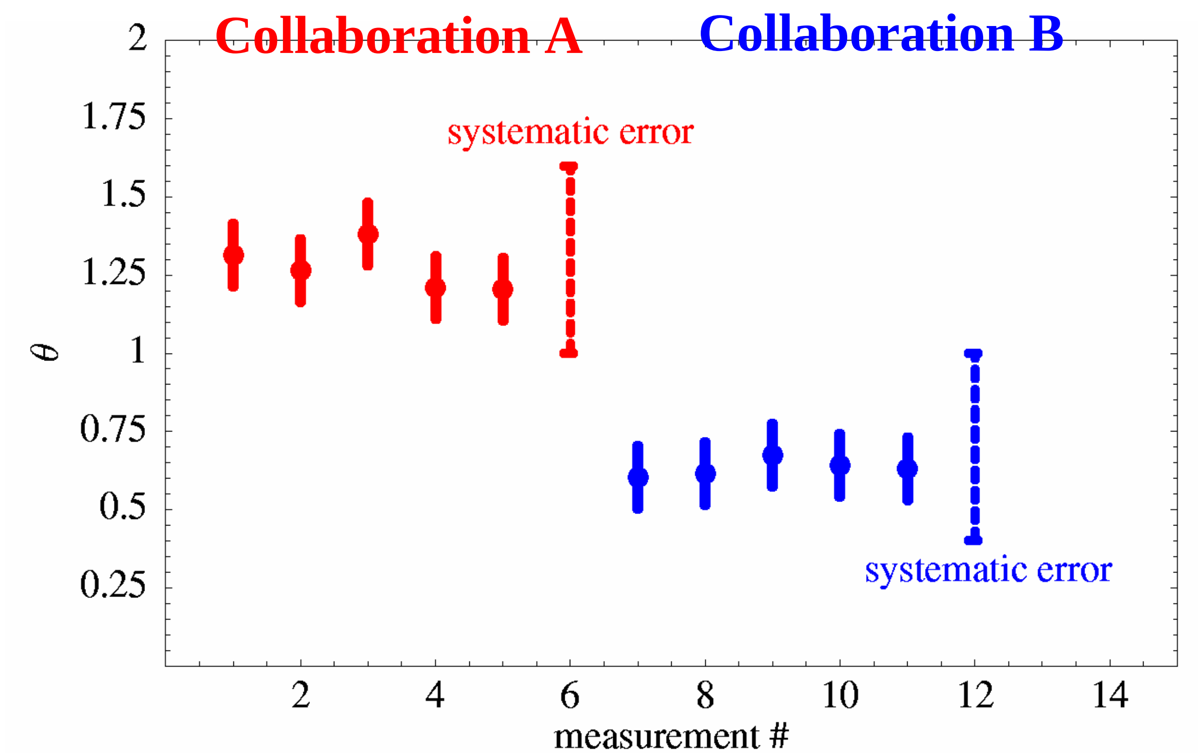
HERA collaborations
ZEUS – S. Chekanov et al; A. Cooper-Sarkar
H1 – C. Adloff et al

The program of Global Analysis is not a routine statistical analysis, because of systematic differences between experiments.

We must sometimes use ***physics judgment*** in this complex real-world problem.

Compatibility

Two experimental collaborations measure the same quantity θ :



The two data sets are consistent within the systematic errors, but there is a systematic difference.

The combined value is a *compromise*, with uncertainty from the systematic errors.

2/
A Study of Compatibility

Table of Data Sets

The PDF's are not exactly CTEQ6 but very close – a no-name generic set of PDF's for illustration purposes.

$$N_{\text{tot}} = 2291$$

$$\chi^2_{\text{global}} = 2368.$$

		N	χ^2	χ^2/N
1	BCDMS F2p	339	366.1	1.08
2	BCDMS F2d	251	273.6	1.09
3	H1 (a)	104	97.8	0.94
4	H1 (b)	126	127.3	1.01
5	H1 (c)	129	108.9	0.84
6	ZEUS	229	261.1	1.14
7	CDHSW F2	85	65.6	0.77
8	NMC F2p	201	295.5	1.47
9	NMC d/p	123	115.4	0.94
10	CCFR F2	69	84.9	1.23
11	E605	119	94.7	0.80
12	E866 pp	184	239.2	1.30
13	E866 d/p	15	5.0	0.33
14	D0 jet	90	62.6	0.70
15	CDF jet	33	56.1	1.70
16	CDHSW F3	96	76.4	0.80
17	CCFR F3	87	26.8	0.31
18	CDF W Lasy	11	8.7	0.79

So, we have accounted for …

- Statistical errors
- Overall normalization uncertainty (by fitting $\{f_{\mathrm{N},e}\}$)
- Other systematic errors (analytically)

We may make further refinements of the fit with weighting factors

$$\chi^2_{\text{global}}(\{a\},\{f_{\mathrm{N}}\}) = \sum_e w_e \chi^2_e(\{a\},\{f_{\mathrm{N}}\}) + \sum_e w_{\mathrm{N},e}\left(\frac{(1-f_e)^2}{\sigma^2_{\mathrm{N},e}}\right)$$

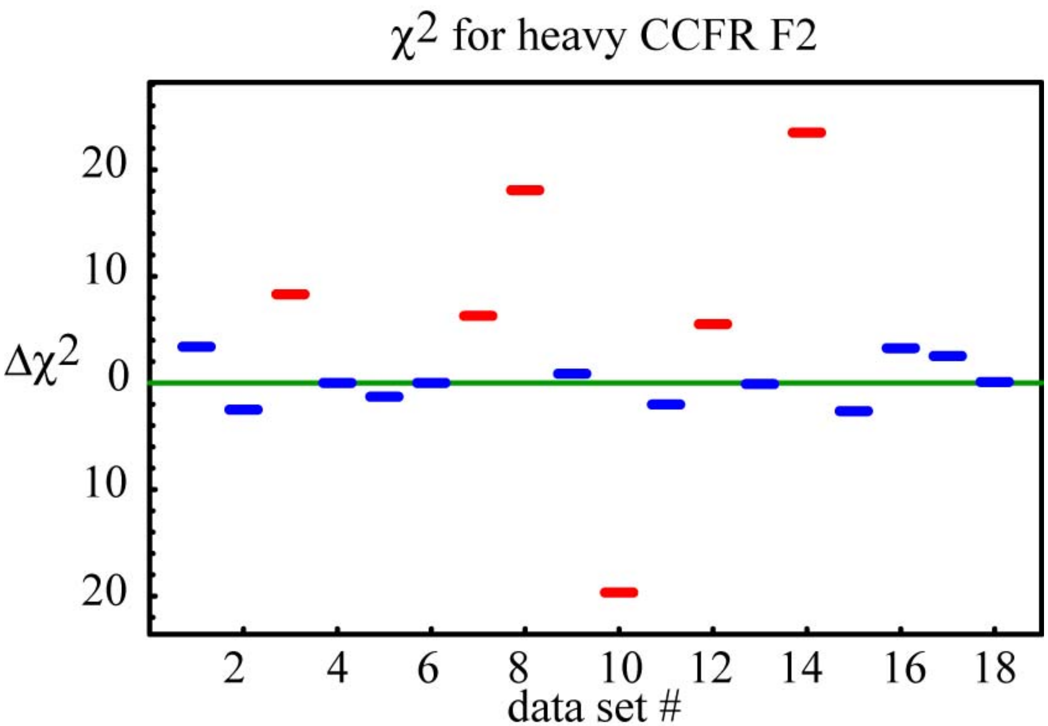
Default : w_e and $w_{\mathrm{N},e} = 1$

The spirit of global analysis is compromise – the PDF's should fit all data sets satisfactorily.

If the default leaves some experiments unsatisfied, we may be willing to reduce the quality of fit to some experiments in order to fit better another experiment. (However, we use this trick sparingly!)

By applying weighting factors in the fitting function, we can test the "compatibility" of disparate data sets.

Example 1.
The effect of giving the CCFR F2 data set a heavy weight.



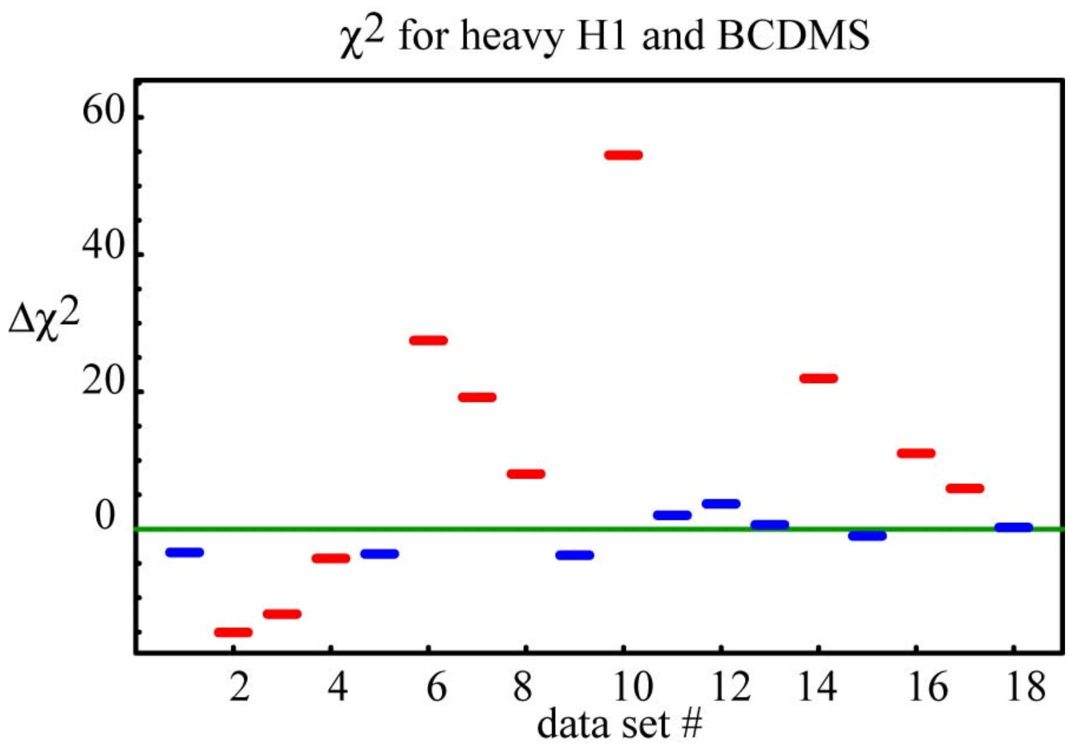
$\Delta\chi^2$

3	H1 (a)	8.3
7	CDHSW F2	6.3
8	NMC F2p	18.1
10	CCFR F2	–19.7
12	E866 pp	5.5
14	D0 jet	23.5

$\Delta\chi^2$ (CCFR) = –19.7
$\Delta\chi^2$ (other) = +63.3

Giving a single data set a large weight is tantamount to determining the PDF's from that data set alone. The result is a significant improvement for that data set but which does not fit the others.

Example 2. Giving heavy weight to H1 and BCDMS



$\Delta\chi^2$

2	BCDMS F2d	–15.1
3	H1 (a)	–12.4
4	H1 (b)	–4.3
6	ZEUS	27.5
7	CDHSW F2	19.2
8	NMC F2p	8.0
10	CCFR F2	54.5
14	D0 jet	22.0
16	CDHSW F3	11.0
17	CCFR F3	5.9

$\Delta\chi^2(\text{H \& B}) = -38.7$

$\Delta\chi^2(\text{other}) = +149.9$

Lessons from these ***reweighting*** studies

• Global analysis requires compromises – the PDF model that gives the best fit to one set of data does not give the best fit to others. This is not surprising because there are systematic differences between the experiments.

• The scale of acceptable changes of χ^2 must be large. Adding a new data set and refitting may increase the χ^2's of other data sets by amounts >> 1.

Clever ways to test the compatibility of disparate data sets

- Plot χ^2 versus χ^2

J Collins and J Pumplin (hep-ph/0201195)

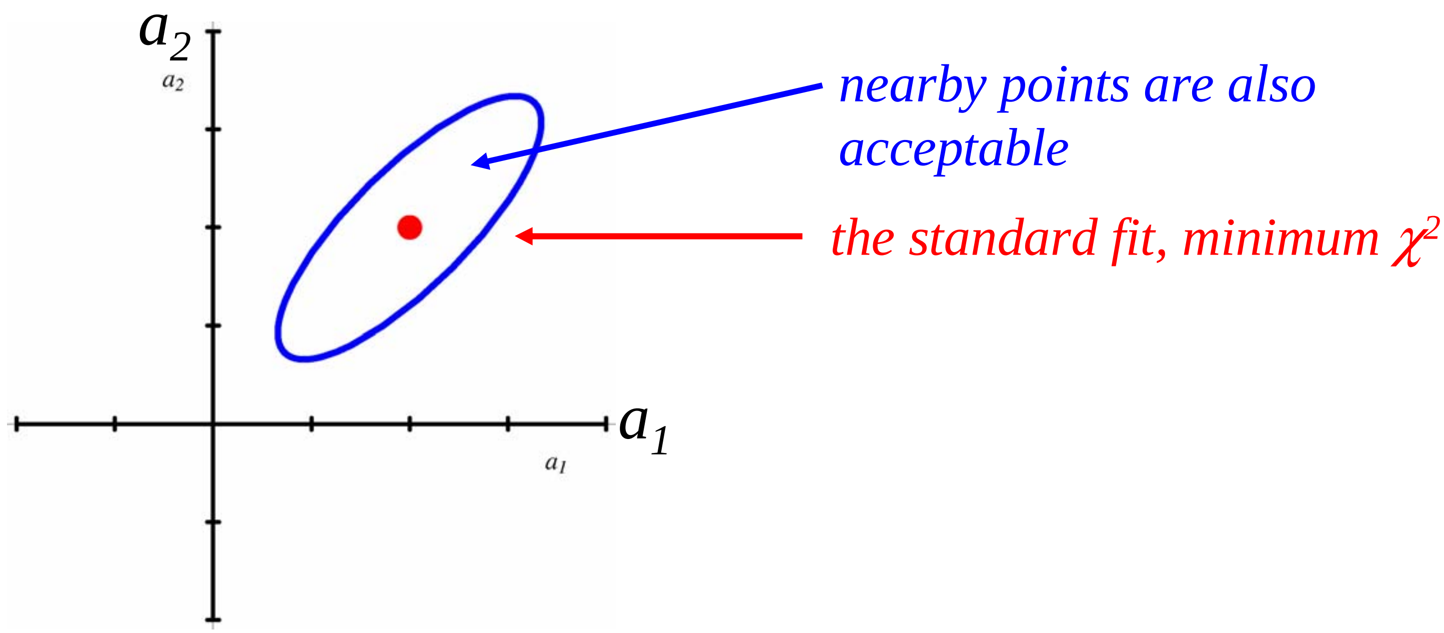
- The Bootstrap Method

Efron and Tibshirani, Introduction to the Bootstrap (Chapman&Hall)

Chernick, Bootstrap Methods (Wiley)

3/
Uncertainty Analysis

We continue to use χ^2_{global} as *figure of merit*. Explore the variation of χ^2_{global} in the neighborhood of the minimum.



The Hessian method

$$H_{\mu\nu} \equiv \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_\mu \partial a_\nu}\bigg|_0 \qquad (\mu, \nu = 1\ 2\ 3 \ldots d)$$

"Master Formula" for the Hessian Method

Classical error formula for a variable $X(a)$

$$(\Delta X)^2 = \Delta\chi^2 \sum_{\mu,\nu} \frac{\partial X}{\partial a_\mu} \left(H^{-1}\right)_{\mu\nu} \frac{\partial X}{\partial a_\nu}$$

Obtain better convergence using eigenvectors of $H_{\mu\nu}$

$$(\Delta X)^2 = \frac{1}{4} \sum_{\mu=1}^{d} \left[X\left(S_\mu^{(+)}\right) - X\left(S_\mu^{(-)}\right) \right]^2$$

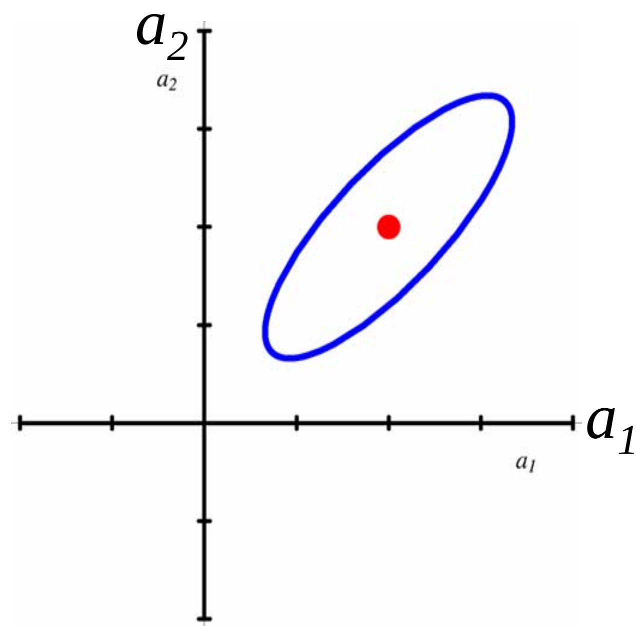
$S_\mu^{(+)}$ and $S_\mu^{(-)}$ denote PDF sets displaced from the standard set, along the ± directions of the μ^{th} eigenvector, by distance $T = \sqrt{(\Delta\chi^2)}$ in parameter space.

Better: Use ***asymmetric*** bounds

The 40 eigenvector basis sets – a complete set of alternate PDFs, tolerably near the minimum of χ^2.

$$\left\{\ S_\mu^{(\pm)}\ ;\ \mu = 1 \cdots d \right\}$$

$$(\Delta X)^2 = \tfrac{1}{4} \sum_{\mu=1}^{d} \left[X\left(S_\mu^{(+)}\right) - X\left(S_\mu^{(-)}\right) \right]^2$$



$S_\mu^{(+)}$ and $S_\mu^{(-)}$ denote PDF sets displaced from the standard set, along the ± directions of the μ^{th} eigenvector, by distance $T = \sqrt{(\Delta\chi^2)}$ in parameter space.

(available in the LHAPDF format : 2d alternate sets)

The Lagrange Multiplier Method

… for analyzing the uncertainty of PDF-dependent predictions.

The fitting function for *constrained fits*

$$F(a_\mu,\lambda) = \chi^2_{\text{global}}(a_\mu) + \lambda X(a_\mu)$$

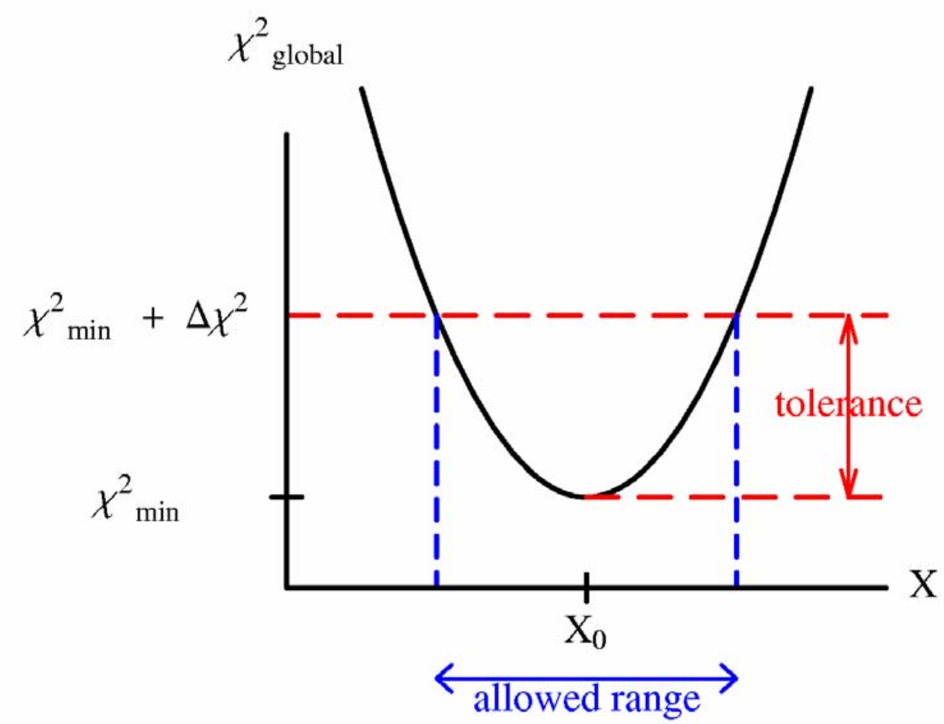
λ : Lagrange multiplier

Minimization of F [w.r.t $\{a_\mu\}$] gives the best fit for the value $X(a_{\text{min},\mu})$ of the variable X.

controlled by the parameter λ

Hence we trace out a curve of χ^2_{global} versus X.

The question of tolerance



X : any variable that depends on PDF's

X_0 : the prediction in the standard set

$\chi^2(X)$: curve of constrained fits

For the specified tolerance ($\Delta\chi^2 = T^2$) there is a corresponding range of uncertainty, $\pm \Delta X$.

What should we use for *T*?

Estimation of parameters in Gaussian error analysis would have

$$T = 1$$

We do not use this criterion.

Aside: The familiar ideal example

Consider N measurements $\{\theta_i\}$ of a quantity θ with normal errors $\{\sigma_i\}$

$$dP = \frac{e^{-r^2/2}}{\sqrt{2\pi}}$$

$$\theta_i = \theta_{\text{true}} + \sigma_i r_i$$

Estimate θ by minimization of χ^2,

$$\chi^2(\theta) = \sum_{i=1}^{N} \frac{(\theta_i - \theta)^2}{\sigma_i^2} \quad \Rightarrow \quad \theta_{\text{combined}} = \frac{\sum_i \theta_i / \sigma_i^2}{\sum_i 1/\sigma_i^2}$$

The mean of θ_{combined} is θ_{true} , the SD is $\Delta\theta_c = \left(\sum_i 1/\sigma_i^2 \right)^{-1/2}$

and

$$\chi^2(\theta_c \pm \Delta\theta_c) - \chi^2(\theta_c) = 1.$$

(= $\sigma / \sqrt{N}$)

The proof of this theorem is straightforward. It does not apply to our problem because of systematic errors.

To judge the PDF uncertainty, we return to the individual experiments.

Lumping all the data together in one variable – $\Delta\chi^2_{global}$ – is too constraining.

Global analysis is a compromise. All data sets should be fit *reasonably* well -- that is what we check. As we vary $\{a_\mu\}$, does any experiment *rule out* the displacement from the standard set?

In testing the goodness of fit, we keep the normalization factors (i.e., optimized luminosity shifts) fixed as we vary the shape parameters.

End result

$$\Delta\chi'^2 \Big|_{\text{fixed norms}} >> 1$$

e.g., ~100 for ~2000 data points.

This does not contradict the $\Delta\chi^2 = 1$ criterion used by other groups, because that refers to a *different* χ^2 in which the normalization factors are continually optimized as the $\{a_\mu\}$ vary.

Some groups *do* use the criterion of $\Delta\chi^2 = 1$ for PDF error analysis.

Often they are using limited data sets – e.g., an experimental group using only their own data. Then the $\Delta\chi^2 = 1$ criterion may underestimate the uncertainty implied by systematic differences between experiments.

An interesting compendium of methods, by R. Thorne

CTEQ6	**$\Delta\chi^2 = 100$ (fixed norms)**
ZEUS	**$\Delta\chi^2 = 50$ (effective)**
MRST01	**$\Delta\chi^2 = 20$**
H1	**$\Delta\chi^2 = 1$**
Alekhin	**$\Delta\chi^2 = 1$**
GKK	**not using χ^2**