Solutions Homework Set 3

Problem 1.7

$$\Psi(x,0) = \begin{cases} A\,x/a & \text{for } 0 \le x \le a \\ A(b-x)/(b-a) & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

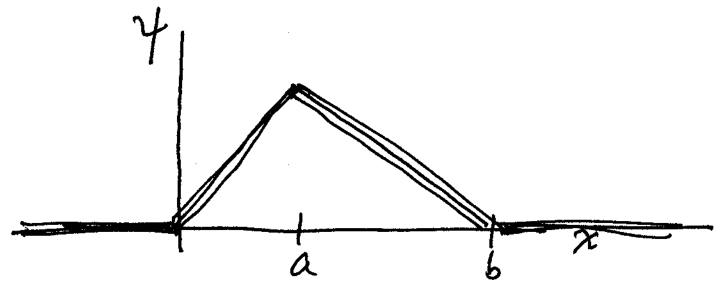
(a) Normalization condition $\int |\Psi|^2 dx = 1$

$$= \int_0^a \left(\frac{Ax}{a}\right)^2 dx + \int_a^b \left(\frac{A(b-x)}{b-a}\right)^2 dx$$

$$= A^2 \left\{ \frac{1}{a^2} \frac{x^3}{3}\Big|_0^a + \frac{1}{(b-a)^2} \frac{(b-x)^3}{3}\Big|_b^a \right\}$$

$$= A^2 \left\{ \frac{a}{3} + \frac{b-a}{3} \right\} = \frac{A^2 b}{3} \implies \boxed{A = \sqrt{\frac{3}{b}}}$$

(b)



(c) Most likely position $= a$

(d) The probability that x is less than a is

$$P(x<a) = \int_0^a |\Psi|^2 dx = \frac{3}{ba^2} \frac{x^3}{3}\Big|_0^a = \frac{a}{b}$$

Checks that this makes sense:

- If $b = a$ then $P(x<a) = 1$ (correct)
- If $b = 2a$ then $P(x<a) = \frac{1}{2}$ (correct)

(e) $\langle x \rangle = \int_a^b |\psi|^2 x \, dx$

$$= \frac{3}{b} \int_0^a \left(\frac{x}{a}\right)^2 x \, dx + \frac{3}{b} \int_a^b \left(\frac{b-x}{b-a}\right)^2 x \, dx$$

1st term $= \frac{3}{b} \frac{1}{a^2} \frac{a^4}{4} = \frac{3a^2}{4b}$

$\boxed{\text{letting } u = b - x}$

2nd term $= \frac{3}{b} \frac{1}{(b-a)^2} \int_0^{b-a} u^2 (b-u) \, du$

$$= \frac{3}{b(b-a)^2} \left\{ b \frac{(b-a)^3}{3} - \frac{(b-a)^4}{4} \right\} = b - a - \frac{3(b-a)^2}{4b}$$

Thus the expectation value of x is

$$\langle x \rangle = \frac{3a^2}{4b} + b - a - \frac{3(b-a)^2}{4b}$$

$$= b - a + \frac{3}{4b} \left[2ab - b^2 \right]$$

$$\boxed{\langle x \rangle = \frac{1}{2} a + \frac{1}{4} b}$$

Checks that the result makes sense:

- If $b = a$ then $\langle x \rangle = \frac{3}{4} a$ OK
- If $b = 2a$ then $\langle x \rangle = a$ OK

Problem 1.8 The wave function is $\Psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t}$

(a) Normalization condition $\int |\Psi|^2 dx = 1$

$$= \int_{-\infty}^{\infty} A^2 e^{-2\lambda|x|} dx = A^2 \int_0^{\infty} e^{-2\lambda x} dx \times 2$$

$$= 2A^2 \frac{1}{2\lambda} e^{-2\lambda x} \Big|_{\infty}^{0} = A^2/\lambda$$

$$\boxed{\therefore A = \sqrt{\lambda}}$$

(b) Expectation values

$$\langle x \rangle = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} x\, dx = 0 \text{ by symmetry}$$

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} x^2 dx$$

$$= 2A^2 \int_0^{\infty} \underbrace{e^{-2\lambda x} x^2}_{= \frac{\partial^2}{\partial\alpha^2} e^{-\alpha x}} dx \quad \text{with } \alpha = 2\lambda$$

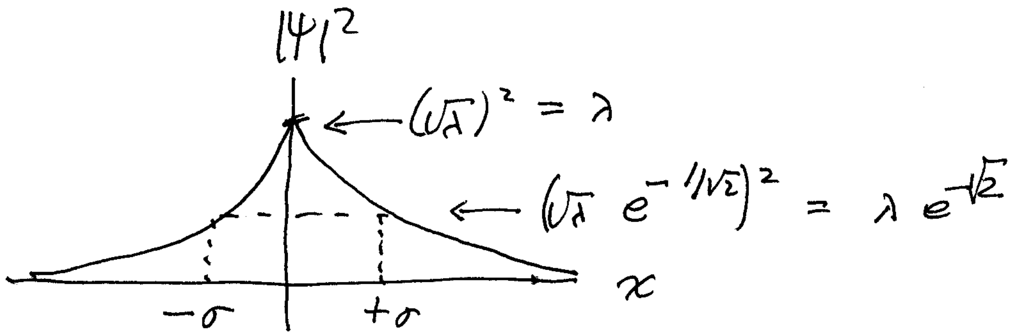
$$= 2A^2 \frac{\partial^2}{\partial\alpha^2} \frac{1}{\alpha} e^{-\alpha x} \Big|_{x=\infty}^{0} = 2A^2 \frac{\partial^2}{\partial\alpha^2}\left(\frac{1}{\alpha}\right)$$

$$= 2A^2 \frac{2}{\alpha^3} \quad \text{with } \alpha = 2\lambda$$

$$= \frac{A^2}{2\lambda^3} = \frac{\lambda}{2\lambda^3} = \frac{1}{2\lambda^2} \qquad \boxed{\langle x^2 \rangle = \frac{1}{2\lambda^2}}$$

(c) The uncertainty in x is

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2}\,\lambda}.$$



The probability that the particle would be found outside $(-\sigma, \sigma)$ is

$$P(|x| > \sigma) = 2\int_{\sigma}^{\infty} |\psi|^2 dx$$

$$= 2\lambda \int_{\sigma}^{\infty} e^{-2\lambda x}\, dx \qquad \text{where } \sigma = \frac{1}{\sqrt{2}\,\lambda}$$

$$= 2\lambda \frac{1}{2\lambda} e^{-2\lambda x} \Big|_{\infty}^{\sigma}$$

$$= e^{-2\lambda\sigma}$$

$$P(|x| > \sigma) = e^{-\sqrt{2}} = 0.243$$

Problem 1.9

The probability that the particle is in the domain (a,b) at time t is

$$P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 \, dx = \int_a^b \rho(x,t) \, dx.$$

Calculate the rate of change of $P_{ab}(t)$

$$\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial \rho}{\partial t} \, dx$$

$$= \int_a^b \left[-\frac{\partial J}{\partial x} \right] dx$$ by the continuity equation

where J = probability current

$$= \frac{\hbar}{2mi} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right)$$

Using the fundamental theorem of calculus, to evaluate the integral,

$$\frac{dP_{ab}}{dt} = -[J(b) - J(a)]$$

$$\boxed{\frac{dP_{ab}}{dt} = J(a,t) - J(b,t)}$$

In words, the rate of change of probability in (a,b) is equal to the current inward at a minus the current outward at b.