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10. Exponential and Logarithm

The exponential function, denoted $\exp x$ or e^x, has many applications in engineering and science. The exponential appears in processes of growth, radioactive decay, electrical resistance, certain probability distributions, and many other places. The inverse function of the exponential is the natural logarithm, denoted $\ln x$ or $\log_e x$. These functions, $\exp x$ and $\ln x$, are encountered so often that every scientific calculator has them built in.[1]

In this chapter we'll study the definitions of $\exp x$ and $\ln x$, learn the derivatives of these functions, and see some physical applications where they are needed. Calculus plays a central role in the study of the exponential and logarithm, as we shall see.

`sec:expfun`

10.1 THE EXPONENTIAL FUNCTION

The exponential function is e^x, a special number e raised to the power x. The domain is all real numbers. There are three puzzling, and fascinating, aspects of this function:

1. What is e?
2. What is meant by "raising e to the power x?"
3. Why is this function so important? What special property does it have?

10.1.1 The number e

The special number e is called *the base of natural logarithms.* It is defined by a limit. The definition is

$$e = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N . \tag{10-1}$$

`eq:defe`

In taking this limit, we may restrict the values of N to integers that become arbitrarily large as $N \to \infty$. This restriction is not strictly necessary, because $(1+1/N)^N$ is a continuous function of N, so that N may be any real number. But if we restrict N to integers, then we avoid the problem of defining irrational powers.[2]

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[1] See Exercise 1.

[2] This subtlety is normally ignored. For any positive integer p, x^p is simple to define ($x^2 = xx$, $x^3 = xxx$, etc.). For fractional p (rational numbers) the definition can easily be

N	$(1+1/N)^N$
1	2
10	2.59374
100	2.70481
1000	2.71692
10^6	2.71828

Table 10.1: The value of $(1+1/N)^N$ approaches e as $N \to \infty$.

The limit in (`eq:defe` 10-1) is a remarkable balancing act. If ξ is a constant number greater than 1, then ξ^N goes to ∞ as $N \to \infty$. But in (`eq:defe` 10-1) the number ξ is $1+1/N$, which approaches 1 as $N \to \infty$, and 1^N is just 1 for any N. Competing tendencies occur as $N \to \infty$. So what is the final value? Naively taking the limit in (`eq:defe` 10-1) we find 1^∞, an undefined quantity. We must work harder to evaluate the limit.

The value of e is approximately 2.718. Table 1 shows numerically how $(1+1/N)^N$ approaches 2.718 as N increases. The numbers in the table were calculated by computer. But can we evaluate e analytically?

We can evaluate e with some help from our old friend, the binomial theorem.[3] In Equation (B-7) let $A = 1$ and $B = 1/N$. The equation gives $(A+B)^N$ in powers of A and B. The result is

$$\left(1+\frac{1}{N}\right)^N = \sum_{k=0}^{N} \left(\frac{1}{N}\right)^k \frac{N!}{k!(N-k)!} \tag{10-2}$$

$$= \sum_{k=0}^{N} \frac{N(N-1)(N-2)\cdots(N-k+1)}{N^k k!}. \tag{10-3}$$

`eq:finsum`

Now take the limit $N \to \infty$. Note that for any k,

$$\frac{N(N-1)(N-2)\cdots(N-k+1)}{N^k} \to 1 \qquad \text{as} \qquad N \to \infty,$$

because the numerator is N^k+"small terms;" the "small terms" become negligible compared to the dominant term N^k as $N \to \infty$. (For example, if $N = 100$ and $k = 2$ then the numerator is 100×99, which is approximately 100^2, the denominator.) Thus the kth term in (`eq:finsum` 10-3) approaches $1/k!$. Also,

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generalized by defining *roots*. But what if p is irrational, like $\sqrt{2}$ or π? Then x^p must be defined by a limiting process.

[3] See Appendix B. The binomial theorem is Equation (B-7).

Table 10.2: Convergence of the series in ($\overset{\texttt{eq:ps}}{\text{10-4}}$). The partial sum means the sum of terms ($1/k!$) from $k = 0$ to $k = K$.

K	$1/K!$	partial sum
0	1	1
1	1	2
2	0.5	2.5
3	0.16666 . . .	2.66666 . . .
4	0.04166 . . .	2.70833 . . .
5	0.00833 . . .	2.7166 . . .
6	0.001389	2.71806
7	0.000198	2.71825
8	0.0000248	2.71828
9	2.76×10^{-6}	2.71828
10	2.76×10^{-7}	2.71828

the sum becomes an infinite series as $N \to \infty$. Finally, then,

$$e = \lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^N = \sum_{k=0}^{\infty} \frac{1}{k!}. \tag{10-4} \quad \boxed{\texttt{eq:ps}}$$

In words, e is the sum of all reciprocal factorials.

An infinite series combines an infinite number of terms. The first 11 terms from ($\overset{\texttt{eq:ps}}{\text{10-4}}$), and the corresponding partial sums, are shown in Table 2. Evidently the added terms are getting very small as k increases, and the sequence of partial sums converges to a number near 2.718.

10.1.2 The function e^x

Next, let's see how to calculate e^x, which is e to the power x. This number can be written as another limit like ($\overset{\texttt{eq:defe}}{\text{10-1}}$). Recall a general theorem on limits: if $A \to A_0$ in some limit then $A^p \to A_0^p$. Applying this to the limit $N \to \infty$ in ($\overset{\texttt{eq:defe}}{\text{10-1}}$), for the power $p = x$, we have

$$\lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^{Nx} = \left[\lim_{N\to\infty} \left(1 + \frac{1}{N}\right)^N\right]^x = e^x. \tag{10-5} \quad \boxed{\texttt{eq:limNp}}$$

Now consider $x > 0$, and change the variable N in the limit on the left side of ($\overset{\texttt{eq:limNp}}{\text{10-5}}$) to $N' \equiv Nx$, which also approaches ∞. So, since $1/N$ is x/N', the limit may be reexpressed as

$$e^x = \lim_{N'\to\infty}\left(1+\frac{x}{N'}\right)^{N'} . \qquad (10\text{-}6) \quad \boxed{\texttt{eq:defex}}$$

(At this point we can drop the prime on N'.) Equation (10-6) can be taken as the definition of e to the power x, even for irrational values of x. We arrived at (10-6) assuming $x > 0$, but it is also correct for $x < 0$.

We may also express e^x as a power series, by again using the binomial theorem in the manner of (10-3),

$$\begin{aligned} e^x &= \lim_{N\to\infty}\sum_{k=0}^{N}\left(\frac{x}{N}\right)^k \frac{N!}{k!(N-k)!} \\ &= \lim_{N\to\infty}\sum_{k=0}^{N}\frac{x^k}{k!}\frac{N(N-1)(N-2)\cdots(N-k+1)}{N^k} \\ &= \sum_{k=0}^{\infty}\frac{x^k}{k!}. \qquad (10\text{-}7) \quad \boxed{\texttt{eq:psx}} \end{aligned}$$

The series expression (10-4) for e is just the special case of (10-7) evaluated at $x = 1$. Note that for $x = 0$ we have $e^0 = 1$, from either (10-6) or (10-7).

The product of exponentials

Theorem 10-1. The product of exponentials is

$$e^a e^b = e^{a+b}. \qquad (10\text{-}8) \quad \boxed{\texttt{eq:prodexp}}$$

Equation (10-8) is a familiar property of a number raised to a power. An analogous equation is

$$10^n 10^m = 10^{n+m}. \qquad (10\text{-}9) \quad \boxed{\texttt{eq:prod10}}$$

For integers n and m, the equation is obvious: 10^n is n factors of 10, 10^m is m factors of 10, and the product is $n + m$ factors of 10. If n and m are fractional, or irrational, the relation is also true but not so easy to prove. How can we prove (10-8) for arbitrary a and b?

Proof. We'll use (10-6) as the definition of e^x. Then

$$e^{a+b} = \lim_{N\to\infty}\left(1+\frac{a+b}{N}\right)^N . \qquad (10\text{-}10)$$

Note that

$$1+\frac{a+b}{N}=\left(1+\frac{a}{N}\right)\left(1+\frac{b}{N}\right)-\frac{ab}{N^2}. \tag{10-11}$$

But in the limit $N\to\infty$, the term ab/N^2 becomes negligible. Therefore we may make the approximation

$$\left(1+\frac{a+b}{N}\right)^N\approx\left(1+\frac{a}{N}\right)^N\left(1+\frac{b}{N}\right)^N, \tag{10-12}$$

valid in the limit of large N; the error approaches 0 as $N\to\infty$. Now recall a general theorem on limits, that $\lim AB=\lim A\times\lim B$; then

$$e^{a+b}=\lim_{N\to\infty}\left(1+\frac{a}{N}\right)^N\times\lim_{n\to\infty}\left(1+\frac{b}{N}\right)^N=e^ae^b, \tag{10-13}$$

and hence the theorem is proven.

Another familiar relation is

$$\left(e^a\right)^c=e^{ac}, \tag{10-14}$$

the proof of which is left as an exercise.[4]

10.1.3 What is so special about e^x?

The answer to the question in the title of this section is related to the *derivative*. We'll prove the following theorem.

Theorem 10-2. The derivative of e^x is e^x.

The equation $\exp'(x)=\exp(x)$, together with the requirement $e^0=1$, determines e^x for all x. The only functions $f(x)$ with the property $f'(x)=f(x)$ are Ce^x where C is a constant. If we demand also $f(0)=1$, then $C=1$ and so $f(x)=e^x$.

Proof. The derivative of e^x is

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to0}\frac{e^x(e^h-1)}{h}\\ &= e^x\lim_{h\to0}\frac{e^h-1}{h}.\end{aligned} \tag{10-15}$$

So the derivative is Ae^x where A is the number

$$A=\lim_{h\to0}\frac{e^h-1}{h}. \tag{10-16}$$

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[4] Exercise `exer:eac` ??.

The theorem is true if $A = 1$. Now use the power series (`eq:psx` 10-7) to evaluate e^h,

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \cdots \ . \tag{10-17}$$

Therefore,

$$\frac{e^h - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots \ . \tag{10-18}$$

`eq:Alim`

In the limit $h \to 0$, the right-hand side of (`eq:Alim` 10-18) approaches 1, because all terms except the first become 0. Hence $A = 1$, and the theorem is proven.

Another proof of the power series

If we take the defining property of $\exp(x)$ to be that $\exp'(x) = \exp(x)$, with $\exp(0) = 1$, then we can derive the power series (`eq:psx` 10-7). First, assume that $\exp(x)$ can be written as a series,

$$\exp(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{k=0}^{\infty} c_k x^k \tag{10-19}$$

`eq:AP1`

where $c_0 = 1$. Then the derivative is

$$\exp'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots = \sum_{k=1}^{\infty} k c_k x^{k-1}. \tag{10-20}$$

`eq:AP2`

Since we require $\exp'(x) = \exp(x)$ for all x, the coefficients for each power of x in (`eq:AP1` 10-19) and (`eq:AP2` 10-20) must be equal. Starting with the power 0 and working up, we find

$$\begin{aligned}
1c_1 &= 1 && \Longrightarrow && c_1 = 1 \\
2c_2 &= c_1 && \Longrightarrow && c_2 = 1/2 = 1/2! \\
3c_3 &= c_2 && \Longrightarrow && c_3 = 1/6 = 1/3! \\
4c_4 &= c_3 && \Longrightarrow && c_4 = 1/24 = 1/4!
\end{aligned}$$

and so on. Evidently, $c_k = 1/k!$. The general term in the sequence is $kc_k = c_{k-1}$, and this recursion relation is satisfied by $c_k = 1/k!$. Thus the power series is

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{10-21}$$

`eq:Taylor`

The power series (`eq:Taylor` 10-21), which converges for all x, is nothing but the Taylor series for the exponential function about $x = 0$. Recall from Chap. 7

the Taylor series expansion for a function $f(x)$ about $x = 0$,

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\,\frac{x^k}{k!} \qquad \text{where} \qquad f^{(k)}(0) = \left.\frac{d^k f}{dx^k}\right|_{x=0} . \tag{10-22}$$

For the exponential function $f(x) = e^x$, every derivative of $f(x)$ is also e^x; thus $f^{(k)}(0) = 1$ for all k. Hence the Taylor series is (`eq:Taylor`10-21).

Summary

We have learned that e^x does have a special property: The derivative of e^x is equal to the value of e^x at every x. Figure `fig:graphexp`?? shows a graph of e^x. At $x = 0$ the value is 1 and the slope is 1. As x increases, the value increases and the slope increases, with value = slope at all x. So the rate of increase increases. This behavior is called *exponential growth.* The function e^x grows faster than any power of x for sufficiently large x.

As x decreases, with $x < 0$, i.e., for x negative, the value of e^x decreases and the slope decreases, so that $e^x \to 0$ as $x \to -\infty$. Equivalently, $e^{-x} \to 0$ as $x \to \infty$. This decrease of e^{-x} with increasing x is called exponential decay, and it is used to describe processes of decay in science and engineering. Some examples of exponential systems are discussed in Sec. `sec:expexample`10.3.

10.2 THE NATURAL LOGARITHM

`sec:natlog`

The natural logarithm is defined as the inverse function of the exponential;[5] that is,

$$\text{If} \qquad y = e^x \qquad \text{then} \qquad \ln y = x. \tag{10-23}$$

The inverse functional relationship may also be expressed as

$$e^{\ln y} = y \qquad \text{or} \qquad \ln\left(e^x\right) = x. \tag{10-24}$$

The domain of the function e^x is all real numbers, $-\infty < x < \infty$. However, e^x is > 0 for any x, so the range of e^x is the positive reals. Therefore the domain of $\ln y$ is $0 < y < \infty$. Figure `fig:ln` ?? shows a graph of $\ln y$ versus y. Recall from Chapter 1 that the graph of an inverse function resembles the graph of the original function with the horizontal and vertical coordinates exchanged. This is the case for Figs. `fig:ln` ?? and `fig:graphexp` ??.

The derivative of $\ln y$ is interesting. The general formula for the derivative of the inverse function $f^{-1}(y)$ of a function $f(x)$ is[6]

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{df/dx} \qquad \text{where} \qquad y = f(x). \tag{10-25}$$

Applying this to the natural log, i.e., for $f^{-1}(y) = \ln y$ and $f(x) = \exp x$,

$$\frac{d}{dy} \ln y = \frac{1}{e^x} = \frac{1}{y}. \tag{10-26}$$

`eq:derlog`

So, in terms of an arbitrary variable ξ,

$$\ln' \xi = \frac{1}{\xi}. \tag{10-27}$$

In words, the derivative of the natural log is the power function with exponent -1. We have seen other functions whose derivative is a power. We learned long ago (Chap. 5) that the derivative of ξ^p is $p\xi^{p-1}$, a power function with exponent $p - 1$. Now consider: What function has derivative ξ^q? If $q \neq -1$ then the answer is $\xi^{q+1}/(q+1)$. But if $q = -1$ then $\xi^{q+1}/(q+1)$ is undefined because of division by 0. If $q = -1$ then the answer is $\ln \xi$ (or, generally, $\ln \xi + C$ where C is a constant).

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[5] Recall that if $y = f(x)$ then $f^{-1}(y) = x$, where f^{-1} denotes the inverse function of f. See Section 1-4.

[6] See Equation (6-xx).

10.2.1 Logarithms with base 10 or other bases

Logarithms are also defined with base numbers other than e. The function $\log_{10} y$ is defined as the inverse function of $y(x) = 10^x$; that is,

$$\text{If} \qquad y = 10^x \qquad \text{then} \qquad x = \log_{10} y. \tag{10-28}$$

In words, $\log_{10}(y)$ is the number x for which 10 to the power x is y. Generally, the base can be any constant a, so that

$$\text{If} \qquad y = a^x \qquad \text{then} \qquad x = \log_a y. \tag{10-29}$$

Base-10 logs have $a = 10$ and natural logs have $a = e$.

All the exponential functions, and their logarithms, are closely related. Consider 10^x. Using the inverse relationship of exp and ln, we may write

$$10 = e^{\ln 10}; \tag{10-30}$$

therefore,

$$10^x = \left(e^{\ln 10}\right)^x = e^{x \ln 10}. \tag{10-31}$$

`eq:10x`

The value of $\ln 10$ is 2.303, so approximately $10^x = e^{2.303x}$. Or, generalizing to an arbitrary base number a,

$$a^x = e^{x \ln a}. \tag{10-32}$$

`eq:ax`

Equation (`eq:ax` 10-32) relates the exponental with arbitrary base a to the *natural* exponential function e^x.

Equation (`eq:ax` 10-32) shows how exponentials with different bases are related. How are the logs related? The inverse function of $y(x) = a^x$ is $x(y) = \log_a y$. By (`eq:ax` 10-32),

$$y = e^{x \ln a} \qquad \text{so} \qquad \ln y = x \ln a. \tag{10-33}$$

We see that $x = \ln y / \ln a$. But $x = \log_a y$; thus the log with base a is proportional to the natural log, and

$$\log_a y = \frac{\ln y}{\ln a}. \tag{10-34}$$

`eq:logax`

Equation (`eq:logax` 10-34) relates the log with base a to the natural log. For example, for base 10 logarithms,

$$\log_{10}(y) = \frac{\ln y}{\ln 10} = 0.4343 \ln y. \tag{10-35}$$

Example 2. What is the derivative of 10^x?

Solution. First, the derivative of e^{Cx}, where C is a constant, is

$$\frac{d}{dx}e^{Cx} = Ce^{Cx}. \tag{10-36}$$

`eq:deCx`

Note how the chain rule has been used. To prove (`eq:deCx` 10-36) we regard e^{Cx} as a composite function,

$$e^{Cx} = \exp\left[g(x)\right] \qquad \text{where} \qquad g(x) = Cx. \tag{10-37}$$

Then by the chain rule,

$$\frac{d}{dx}e^{Cx} = \frac{d\exp(g)}{dg}\frac{dg}{dx} = \exp(g)C = Ce^{Cx}. \tag{10-38}$$

Now apply the rule (`eq:deCx` 10-36) to the function 10^x, as expressed in (`eq:10x` 10-31),

$$\frac{d}{dx}10^x = e^{x\ln 10}\ \ln 10 = 10^x\ \ln 10. \tag{10-39}$$

`eq:d10x`

Generalization Following the same line of argument that gives (`eq:d10x` 10-39), for an arbitrary base number a,

$$\frac{d}{dx}a^x = e^{x\ln a}\ \ln a = a^x\ \ln a. \tag{10-40}$$

`eq:gende`

Equation (`eq:gende` 10-40) is a general result—the derivative of the general exponential function a^x with arbitrary base a. For the special case $a = e$ we have the natural exponential; then, because $\ln e = 1$,

$$\frac{d}{dx}e^x = e^x,$$

which is again Theorem 2.

The exponential derivatives (`eq:deCx` 10-36) and (`eq:gende` 10-40) are recorded in the Table of Derivatives in Appendix E, as is the derivative of the log (`eq:derlog` 10-26). These derivatives are worth knowing because the exponential function is often encountered in applied mathematics. Please commit them to memory!

`sec:expexample`

10.3 EXAMPLES

Example 3. Let $P(v) = v^2 e^{-v^2}$ for $v \geq 0$. What is the derivative dP/dv? Describe the functions $P(v)$ and $P'(v)$. Where is the maximum of $P(v)$?

Solution. To calculate the derivative we must apply both the Leibniz rule and the chain rule. $P(v)$ is a product of v^2 and e^{-v^2}; the second function is a composite function $g\left[h(v)\right]$ where $g(\xi) = e^{-\xi}$ and $h(v) = v^2$. By the Leibniz Rule,

$$\frac{dP}{dv} = \left(\frac{dv^2}{dv}\right) e^{-v^2} + v^2 \frac{d}{dv}\left(e^{-v^2}\right). \tag{10-41}$$

`eq:P1`

In the first term, $dv^2/dv = 2v$. In the second term we must differentiate e^{-v^2} by the chain rule,

$$\begin{aligned} \frac{d}{dv}\left(e^{-v^2}\right) &= \frac{dg}{d\xi}\frac{d\xi}{dv} \quad \text{letting} \quad g(\xi) = e^{-\xi} \text{ and } \xi = v^2, \\ &= \left(-e^{-\xi}\right)(2v) = -2ve^{-v^2}. \end{aligned} \tag{10-42}$$

`eq:P2`

Combining (`eq:P1` 10-41) and (`eq:P2` 10-42),

$$\frac{dP}{dv} = 2ve^{-v^2} - 2v^3 e^{-v^2} = 2v(1 - v^2)e^{-v^2}. \tag{10-43}$$

Figure `fig:EXMax` ?? shows graphs of $P(v)$ and $P'(v)$. From Fig. `fig:EXMax` ??(a) note these properties of $P(v)$:

(i) the slope is 0 at $v = 0$;

(ii) the slope is > 0 for $0 < v < 1$;

(iii) the slope is 0 at $v = 1$, where $P(v)$ is maximum;

(iv) the slope is < 0 for $v > 1$;

(v) the slope $\to 0$ as $v \to \infty$.

The derivative $P'(v)$ is the slope of $P(v)$, and Fig. `fig:EXMax` ??(b) agrees with the properties (i)–(v).

Example 4. The probability distribution of molecular speeds in a gas at temperature T is

$$P(v) = Cv^2 \exp\left(-mv^2/2kT\right) \quad \text{where} \quad C = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}. \tag{10-44}$$

`eq:MaxBol`

This function, which resembles the function in Example 3, is called the Maxwell-Boltzmann distribution. (m is the molecular mass, and k is Boltzmann's constant, $1.38 \times 10^{-23}\,\mathrm{J/K}$.) The speed where $P(v)$ is maximum is the most probable speed for a molecule. Where is the maximum of $P(v)$? What is the most probable speed of a nitrogen molecule in air at room twemperature?

Solution. We may write $P(v)$ in the form

$$P(v) = Cv^2 e^{-\gamma v^2} \quad \text{where} \quad \gamma = \frac{m}{2kT}. \tag{10-45}$$

The parameters γ and C are constant with respect to the variable v. The maximum occurs where $dP/dv = 0$; that is,

$$\begin{aligned} C2ve^{-\gamma v^2} + Cv^2\left(-2\gamma v\right)e^{-\gamma v^2} &= 0, \\ 2Cv\left(1 - \gamma v^2\right)e^{-\gamma v^2} &= 0. \end{aligned} \tag{10-46}$$

The most probable speed is

$$v_{\mathrm{mp}} = \frac{1}{\sqrt{\gamma}} = \sqrt{\frac{2kT}{m}}. \tag{10-47}$$

For example, for nitrogen molecules (N_2) in air at room temperature, $T = 20\,\mathrm{C} = 293\,\mathrm{K}$, the most probable speed is

$$v_{\mathrm{mp}} = \sqrt{\frac{2 \times 1.38 \times 10^{-23}\mathrm{J/K} \times 293\mathrm{K}}{28 \times 1.66 \times 10^{-27}\mathrm{kg}}} = 420\,\mathrm{m/s}. \tag{10-48}$$

Example 5. **Radioactive carbon dating**

^{14}C is a radioactive isotope of carbon. (The most common stable isotope is ^{12}C.) It decays by the process

$$^{14}\mathrm{C} \longrightarrow {}^{14}\mathrm{N} + \mathrm{e} + \bar{\nu},$$

with a half-life of $T_{1/2} = 5370\,\mathrm{y}$. ^{14}C is produced in the atmosphere by cosmic rays, and is absorbed as CO_2 by living plants. When the plant dies, it no longer absorbs carbon and its ^{14}C decays. The proportion of ^{14}C in a plant sample can be used to determine the age of the sample, i.e., the time that has elapsed since the plant was living.

Suppose the fraction of the initial ^{14}C that remains in a sample today is x. That is, $N/N_0 = x$ where N = number of ^{14}C atoms today and N_0 = number of ^{14}C atoms when the plant was living. Then what is the age of the sample? In particular, what is the age of the sample if $x = 0.1$?

Solution. The half-life is $T_{1/2} = 5730\,\mathrm{y}$. One-half of the ^{14}C present at any time will decay during the next half-life. So, after n half-lives since the death of the plant, the remaining number of ^{14}C atoms is

$$N = N_0 \left(\frac{1}{2}\right)^n . \tag{10-49}$$

`eq:N=`

(After one half-life, the number of remaining ^{14}C atoms is $N_0/2$; after two half-lives the remaining number is $N_0/4$; etc.) In terms of the time t since the plant's death ($t = nT_{1/2}$) the fraction x may be written as

$$x = \frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/T_{1/2}} = 2^{-t/T_{1/2}}. \tag{10-50}$$

Or, we may rewrite x in terms of the exponential function, using (`eq:ax` 10-32), as

$$x = e^{-Rt} \qquad \text{where} \qquad R = \frac{\ln 2}{T_{1/2}}. \tag{10-51}$$

`eq:x=e-Rt`

In words, x decays exponentially. To solve for the age t of the sample, take the natural log of both sides of (`eq:x=e-Rt` 10-51). Note that $\ln x = -Rt$, so

$$t = -\frac{1}{R}\ln x = \frac{-\ln x}{\ln 2} T_{1/2}. \tag{10-52}$$

For example, if $x = 0.1$ then the age of the sample is $t = 19,000\,\mathrm{y}$.

Example 6. **Spirals.** Figure `fig:spiral` ?? shows a spiral curve. A point on the curve may be specified by polar coordinates—angular coordinate θ and radial coordinate r. The spiral is traced out by a moving point for which θ increases linearly with time and r increases at a rate proportional to r. Then what is the function $r(\theta)$ for the spiral?

Solution. Since θ increases linearly with time, write $\theta = \omega t$ where $t =$ time and $\omega =$ constant angular rate. The rate of growth of r is

$$\frac{dr}{dt} = \alpha r \tag{10-53}$$

`eq:DE`

where α is a constant. The solution of this differential equation is[7]

$$r(t) = r_0 e^{\alpha t}. \tag{10-54}$$

`eq:solDE`

In words, r grows exponentially. The constant r_0 is the radius at time $t = 0$.

---

[7] Please verify that the function (`eq:solDE` 10-54) does satisfy the differential equation (`eq:DE` 10-53).

The solution to the example is obtained by writing r as a function of θ,

$$r(\theta) = r_0 e^{\theta\alpha/\omega}. \tag{10-55}$$

Figure `fig:spiral`?? shows an example of the spiral, with $r_0 = 1$ and $\omega/\alpha = \tan(70^\circ)$.

The curve in Fig. `fig:spiral`?? is called a logarithmic spiral. It bears a resemblance to the shells of some snails, and that of the chambered nautilus (*Nautilus Nautilus*). The exponential function arises naturally (i.e., in nature) when a rate of growth is proportional to the size of the growing system.

Another name for this curve is the *equiangular spiral*, because all radial lines from the origin cross the spiral at the same angle β, given by

$$\beta = \arctan \frac{\omega}{\alpha}. \tag{10-56}$$

The equiangular spiral has many symmetries, which were discovered by Jacob Bernoulli.[8] He was so impressed by this curve that he had it carved on his tombstone.

Example 7. **Electrical resistance**

Figure `fig:circuit`??(a) shows a capacitor C (two parallel conducting plates) connected to a 3-volt battery. When fully charged the plates have charge $+Q_0 = CV_0$ and $-Q_0$. At time $t = 0$ the switch is moved to the other position. Then the light bulb lights as current $I(t)$ flows from the positive plate to the negative plate. Determine the function $I(t)$. Assume $C = 1\,$farad (very large!) and the resistance of the bulb filament is $R = 2\,$ohms.

Solution. Let $Q(t)$ be the charge on the positive plate. Current is the rate of flow of charge, so the current is

$$I = -\frac{dQ}{dt}. \tag{10-57}$$

(As the capacitor charge Q decreases, current flows downward through the light bulb.) The potential difference across the capacitor is Q/C, and this must equal the potential difference across the resistance, which is IR by Ohm's law. Thus

$$-R\frac{dQ}{dt} = \frac{Q}{C}, \qquad \text{or} \qquad \frac{dQ}{dt} = -\frac{1}{RC}Q. \tag{10-58}$$

The solution to this differential equation is

$$Q(t) = Q_0 e^{-t/RC} \tag{10-59}$$

---

[8] Jacob Bernoulli was a contemporary of Newton and Leibniz, and contributed to the early development of calculus.

where $Q_0 = CV_0$ is the initial charge.[9] The current as a function of time is

$$I(t) = -\frac{dQ}{dt} = \frac{V_0}{R} e^{-t/RC}. \tag{10-60}$$

Figure `fig:circuit` ??(b) shows the current as a function of time. The current decays exponentially. Note that I decreases by a factor of $1/e = 0.368$ during any time interval of $\Delta t = RC$, because

$$I(t + \Delta t) = \frac{V_0}{R} e^{-(t+\Delta t)/RC} = \frac{1}{e} I(t).$$

This *decay time* is

$$RC = 2\,\mathrm{ohms} \times 1\,\mathrm{farad} = 2\,\mathrm{sec}. \tag{10-61}$$

The very large capacitance (1 farad) implies a slow decay. In everday experience a light bulb turns off instantly because the capacitance in the circuit is small. (In an ordinary lamp the decay time is determined by the thermal conductivity rather than the capacitance, because the time constant RC is much less than the thermal decay time.)

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[9] Please verify that the differential equation and initial condition are satisfied.

10.4 THE HISTORY OF LOGARITHMS

`sec:hist`

Historically, logarithms were developed as a computational technique.

The first system of *logarithms* was invented by John Napier. He was not a professional mathematician, but a Scottish laird (the Baron of Murchiston). He coined the word 'logarithm,' and published a description of his system entitled *Mirifici logarithmorum canonis descriptio* in 1614. Napier's goal was to provide a simple but accurate method for computing products and quotients of numbers. Let x be a real number; Napier's log of x, denoted here by ℓ_x, was defined by

$$x = 10^7 \left(1 - 10^{-7}\right)^{\ell_x} .$$

By making a table of x and ℓ_x values, multiplication of numbers could be reduced to addition of the logs and table look up. Consider

$$xy = 10^7 \; 10^7 \left(1 - 10^{-7}\right)^{\ell_x + \ell_y} ;$$

the product of x and y is 10^7 times the antilog of $\ell_x + \ell_y$. The idea of logarithms was adopted widely for scientific work at that time because it greatly simplified numerical computations. For example, Kepler used logarithms in analyzing the data on planetary observations, from which he deduced the laws of planetary motion.

The Napierian log is different from the natural log, but approximately related because

$$\ell_x = \frac{\ln\left(10^{-7}x\right)}{\ln\left(1 - 10^{-7}\right)} \approx -10^7 \ln\left(\frac{x}{10^7}\right) .$$

The computational idea was modified by Henry Briggs, a professor of geometry at Oxford. He introduced a base number and defined ℓ_x as the power that would give x. Briggs chose 10 as his base number, and replaced Napier's definition of ℓ_x by

$$x = 10^{\ell_x};$$

that is, $\ell_x = \log_{10} x$. The logarithms of Briggs were called 'common logarithms.'

The natural logarithm, with base $e \equiv \lim_{N\to\infty} \left(1 + 1/N\right)^N$, is important today not as a computational tool but because the exponential function is the solution to a very basic differential equation,

$$\begin{cases} \dfrac{df}{dx} = f(x) \\ \text{with} \quad f(0) = 1 \end{cases} \qquad \Longrightarrow \qquad f(x) = e^x .$$

This and related equations arise in many applications in science and engineering. The natural log is the inverse function of e^x. The universal notation of e for the base of natural logs was created by the great Leonhard Euler.

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