

## 13. The Integral

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The *concept* of integration, and the motivation for developing this concept, were described in the previous chapter. Now we must define *the integral*, carefully and completely.

According to Webster's dictionary, "to integrate" means to make complete by adding parts. This is indeed what is done in integration in calculus. The integral is the resulting total.

### 13.1 DEFINITIONS, TERMINOLOGY, AND NOTATIONS

We must begin by establishing some terminology.

**Region of integration.** The region of integration is a segment  $[a, b]$  of the real line. That is, the region is the set of points  $x$  with  $a \leq x \leq b$ .

**Differential elements.** The region of integration will be subdivided into many small parts, as illustrated in Fig. [fig:diffe1ts](#). These small segments of the region  $[a, b]$  are called the differential elements, or just elements. There are in general  $N$  elements, each of length  $\Delta x$ . Note that  $N\Delta x$  is the length of the full region of integration, i.e.,  $b - a$ , so

$$\Delta x = \frac{b - a}{N}. \tag{13-1}$$

The set of points used to subdivide the range  $[a, b]$  (see Fig. [fig:diffe1ts](#)) is

$$\{x_0, x_1, x_2, x_3, \dots, x_{N-1}, x_N\}$$

where  $x_0 = a$  is the left endpoint and  $x_N = b$  is the right endpoint. We denote a generic point in this set by  $x_k$ , where the index  $k$  has an integer value from 0 to  $N$ . Therefore the position on the real line of the point  $x_k$  is

$$x_k = a + k(\Delta x). \tag{13-2}$$

For example,  $x_1 = a + \Delta x$ ,  $x_2 = a + 2\Delta x$ , etc. Note that the right-most point is

$$x_N = a + N\Delta x = a + (b - a) = b,$$

i.e., the right endpoint of the full range. The  $k$ th differential element is the small segment  $[x_{k-1}, x_k]$  whose length is  $x_k - x_{k-1} = \Delta x$ .

The integral will be defined as a certain sum of terms, in the limit  $\Delta x \rightarrow 0$  and  $N \rightarrow \infty$ . In taking the limit,  $N \Delta x$  remains constant, equal to  $b - a$ .

**The definite integral.** The definite integral<sup>1</sup> of a function  $g(x)$  for the integration region  $[a, b]$  is the sum of terms  $g(x_k)\Delta x$  for the subdivision shown in Fig. 13-3, in the limit  $\Delta x \rightarrow 0$ . Written as an equation,

$$\text{the definite integral} = \lim_{\Delta x \rightarrow 0} \sum_{k=1}^N g(x_k)\Delta x.$$

Here  $\sum_{k=1}^N$  indicates that we sum over the  $N$  elemental segments, of which the  $k$ th is  $[x_{k-1}, x_k]$ . The mathematical symbol for a discrete sum is  $\Sigma$  (the Greek letter with the sound of ‘S’). The  $\Sigma$  notation for summation is described in Appendix B.

### 13.1.1 Notation for the definite integral

The mathematical symbol for the integral is  $\int$ . It resembles an elongated ‘S’, which is appropriate because it stands for a **S**ummation of many parts.

A definite integral depends on three things, and the notation for the integral must display all this information. First, there is the function that is being integrated, which is called the *integrand*; let  $g(x)$  denote the function. Second, there is the variable of integration, which is the independent variable  $x$  of the function  $g(x)$ . Third, there is the region (or *range*) of integration  $[a, b]$ ; this is the set of real numbers  $x$  with  $a \leq x \leq b$ . The notation for the definite integral is

$$\int_a^b g(x)dx. \quad (13-3) \quad \boxed{\text{eq:intnot}}$$

Note how it displays the three pieces of information: the function is  $g(x)$ , the independent variable ( $x$ ) is indicated in  $dx$ , and the endpoints of the range of integration are  $a$  and  $b$ . The expression in (13-3) is read as “the integral of  $g(x)$  from  $a$  to  $b$ .”

The complex symbol in (13-3) stands for a single number. For example, the integral of the function  $x^2$  for  $x$  from 1 to 3 is denoted by

$$\int_1^3 x^2 dx;$$

<sup>1</sup>We define in this chapter the “definite” integral, which must have a specified region of integration. The “indefinite” integral will be defined in Chapter 14.

the *value* of this integral, which we'll learn to calculate in Chapter 14, is  $26/3$ . Thus we may write an equation,

$$\int_1^3 x^2 dx = \frac{26}{3} = 8.666\dots$$

### 13.1.2 Operational definition of the integral

Summarizing all that has been said, the equation that defines the definite integral of a function  $g(x)$  over the range  $a \leq x \leq b$  is

$$\int_a^b g(x) dx = \lim_{\Delta x \rightarrow 0} \sum_{k=1}^N g(x_k) \Delta x. \quad (13-4) \quad \boxed{\text{eq:defintdef}}$$

The set of points  $\{x_0, x_1, x_2, \dots, x_N\}$  is specified in Fig. 13-7. The sum on the right-hand side of (13-4) is called the *Riemann sum*.

Note, in (13-3) and (13-4), the appearance of our old friend the differential  $dx$ . A good notation simplifies mathematics. The clever idea behind the notation in (13-3) is that  $g(x)dx$  is the “infinitesimal” contribution to the total from the differential element  $dx$ . The integral is the sum of all the elemental parts.

If we think of  $g(x)$  as the *density* of a quantity  $Q$ , i.e., the amount of  $Q$  per unit of  $x$ , then  $g(x)dx$  is the amount of  $Q$  in the infinitesimal segment  $dx$  of the  $x$  axis. The integral (13-4) is the total of  $Q$  over the integration region  $[a, b]$ .

In principle, (13-4) could be used to evaluate any integral, by numerical computation of the sum for very small values of  $\Delta x$ . Indeed, this method is used to evaluate integrals by computer. We'll consider some simple examples of this operation in the next section. However, doing the calculation this way is very cumbersome in practice. We will develop more powerful techniques to calculate integrals in Chapters 14 to 17. But in any case (13-4) is the mathematical definition of the integral.

## 13.2 EXAMPLES

We will evaluate here some simple integrals directly from the definition (13-4). The sum on the right-hand side of (13-4) is called a *Riemann sum*, named for the mathematician Riemann.<sup>2</sup>

Evaluating the Riemann sum is a very tedious way to calculate an integral. We will discover a more powerful method in Chapter 14. But it is instructive to look at some very simple integrals by using (13-4) directly, to understand the basic idea of integration.

**Example 1.** Consider the constant function  $y(x) = 3.6$ . Determine the definite integral of  $y(x)$  over the integration range  $2 \leq x \leq 7$ .

**Solution.** Subdivide the region of integration  $[2, 7]$  into  $N$  elements of length  $\Delta x = 5/N$ . The Riemann sum is a sum of  $N$  terms, and the contribution from the elemental segment  $[x_{k-1}, x_k]$  is

$$y(x_k)\Delta x = 3.6 \times \frac{5}{N}. \quad (13-5)$$

This elemental contribution is independent of  $k$  because  $y(x)$  is constant, so the Riemann sum is a sum of equal terms, i.e., just equal to  $N$  times any one term,

$$\sum_{k=1}^N y(x_k)\Delta x = 3.6 \times \frac{5}{N} \times N = 18. \quad (13-6)$$

In this example—a constant function—the limit  $\Delta x \rightarrow 0$  (or, equivalently,  $N \rightarrow \infty$ ) is trivial because the Riemann sum does not depend on  $\Delta x$ . The limit of a constant is the constant! Hence the integral is

$$\int_2^7 3.6 \, dx = 18. \quad (13-7)$$

**Generalization.** The example of a constant function is so simple that we can immediately generalize Example 1 to an arbitrary constant function, say  $g(x) = C$ , and an arbitrary range of integration,  $a \leq x \leq b$ . We'll prove that

$$\int_a^b C \, dx = C(b - a). \quad (13-8) \quad \boxed{\text{eq:intC}}$$

Note that the answer to Example 1 agrees with this general formula.

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<sup>2</sup>Bernhard Riemann, who lived from 1826 to 1866, clarified the proper definition of the integral. Another of his developments—Riemannian geometry—is the essential mathematics used in Einstein's theory of general relativity.

To prove the general formula, repeat the analysis in Example 1. In the general case the point  $x_k$  in the subdivision is  $x_k = a + k(\Delta x)$  where  $\Delta x = (b - a)/N$  is the width of an elemental segment. Then the definition of the integral gives

$$\int_a^b C dx = \lim_{N \rightarrow \infty} \sum_{k=1}^N C \Delta x = \lim_{N \rightarrow \infty} C \Delta x N. \quad (13-9)$$

Substituting  $N\Delta x = b - a$  we find the general result

$$\int_a^b C dx = C(b - a). \quad (13-10) \quad \boxed{\text{eq:intCa}}$$

The limit  $N \rightarrow \infty$  is trivial in this case, because the Riemann sum does not depend on  $N$ .

**Example 2.** Consider the function  $f(\xi) = 3\xi$  where the independent variable is denoted by  $\xi$ . Determine the definite integral of  $f(\xi)$  from  $\xi = 0$  to  $\xi = 10$ .

**Solution.** Subdivide the integration region  $[0, 10]$  on the real line of  $\xi$ , into  $N$  segments of width  $\Delta\xi = 10/N$ . The endpoints of the elements are points in the set  $\{\xi_0, \xi_1, \xi_2, \dots, \xi_N\}$  where  $\xi_k = k(\Delta\xi)$ . (Note that  $\xi_0 = 0$  and  $\xi_N = 10$ .) The contribution to the Riemann sum from the  $k$ th segment is

$$f(\xi_k)\Delta\xi = 3\xi_k\Delta\xi = 3k(\Delta\xi)^2. \quad (13-11)$$

The Riemann sum for  $N$  elements, call it  $R_N$ , is

$$R_N = \sum_{k=1}^N 3k(\Delta\xi)^2 = 3(\Delta\xi)^2 \sum_{k=1}^N k; \quad (13-12)$$

in the second equality we have factored out the constant factors 3 and  $(\Delta\xi)^2$ . Recall that the sum of the first  $N$  integers is<sup>3</sup>

$$\sum_{k=1}^N k = 1 + 2 + 3 + \dots + N = \frac{1}{2}N(N + 1). \quad (13-13)$$

Thus

$$R_N = \frac{3}{2}N(N + 1)(\Delta\xi)^2, \quad (13-14)$$

or, substituting  $\Delta\xi = 10/N$ ,

$$R_N = \frac{3}{2}N(N + 1)\frac{100}{N^2} = 150 + \frac{150}{N}. \quad (13-15)$$

<sup>3</sup>See Appendix B.

Now, to obtain the integral we must take the limit  $\Delta\xi \rightarrow 0$ , which is equivalent to  $N \rightarrow \infty$  because  $\Delta\xi = 10/N$ . The term  $150/N$  goes to 0 in the limit, so the integral is

$$\int_0^{10} 3\xi d\xi = 150. \quad (13-16)$$

We'll generalize this result after the next example.

**Example 3.** Consider again  $f(\xi) = 3\xi$ , but now integrate the function from  $\xi = 5$  to  $\xi = 10$ .

By the definition (eq:defintdef),

$$\int_5^{10} 3\xi d\xi = \lim_{\Delta\xi \rightarrow 0} \sum_{k=1}^N 3\xi_k (\Delta\xi). \quad (13-17)$$

In this case  $\Delta\xi$  is  $5/N$  because the region of integration has length 5 and is subdivided into  $N$  elements. Also,  $\xi_k$  is the value of  $\xi$  on the real line at the right side of the  $k$ th element,

$$\xi_k = 5 + k\Delta\xi. \quad (13-18)$$

For example  $\xi_1$  is at  $5 + \Delta\xi$ ,  $\xi_2$  is at  $5 + 2\Delta\xi$ , and so on;  $\xi_N$  is at  $5 + N\Delta\xi = 10$  as required. The Riemann sum for a subdivision with  $N$  elements is

$$\begin{aligned} R_N &= 3 \sum_{k=1}^N (5 + k\Delta\xi) \Delta\xi \\ &= 3 \left[ 5N + \frac{1}{2}N(N+1)\Delta\xi \right] \Delta\xi \\ &= 15 \times 5 + \frac{75N(N+1)}{2N^2} \end{aligned} \quad (13-19)$$

where in the last step we have substituted  $\Delta\xi = 5/N$ . Finally, we obtain the value of the integral by taking the limit  $N \rightarrow \infty$ . As in the previous example,  $N(N+1)/N^2 \rightarrow 1$ . Hence the integral is

$$\int_5^{10} 3\xi d\xi = 75 + \frac{75}{2} = \frac{225}{2}. \quad (13-20)$$

**Generalization.** We may generalize the results of Examples 2 and 3 to any linear function of the form  $f(x) = mx$  with  $m$  a constant, integrated between arbitrary points  $x = a$  and  $x = b$ . We'll prove that

$$\int_a^b mx dx = \frac{m}{2} (b^2 - a^2). \quad (13-21) \quad \boxed{\text{eq:intmx}}$$

Note that the result of Example 3 agrees with this general formula, for  $m = 3$ ,  $a = 5$  and  $b = 10$ .

To prove the general formula, repeat the analysis in Example 3. In the general case the point  $x_k$  in the subdivision is  $x_k = a + k(\Delta x)$  where  $\Delta x = (b - a)/N$  is the width of an elemental segment. Then the definition of the integral gives

$$\begin{aligned} \int_a^b m x dx &= \lim_{N \rightarrow \infty} \sum_{k=1}^N m(a + k\Delta x) \Delta x \\ &= \lim_{N \rightarrow \infty} m \left[ Na\Delta x + \frac{N(N+1)}{2} (\Delta x)^2 \right]. \end{aligned} \quad (13-22)$$

Now substitute  $N\Delta x = b - a$  and take the limit  $N \rightarrow \infty$ :

$$\begin{aligned} \int_a^b m x dx &= \lim_{N \rightarrow \infty} m \left[ a(b - a) + \frac{1}{2} \frac{N(N+1)}{N^2} (b - a)^2 \right] \\ &= m \left[ a(b - a) + \frac{1}{2} (b - a)^2 \right] \\ &= m \left[ a + \frac{1}{2} (b - a) \right] (b - a) \\ &= m \left( \frac{b + a}{2} \right) (b - a) = \frac{m}{2} (b^2 - a^2). \end{aligned} \quad (13-23)$$

In this calculation the limit  $N \rightarrow \infty$  was evaluated in the second step by replacing  $N(N + 1)/N^2$  by 1.

### 13.2.1 The integral of an arbitrary linear function between arbitrary endpoints

Equations [\(13-8\)](#) and [\(13-21\)](#) are nice, general formulas. But we can do better. In this section we'll determine the formula for the integral of *any linear function*.

A linear function of the independent variable  $x$  has the form  $C_0 + C_1x$  where  $C_0$  and  $C_1$  are constants. The function  $y(x) = 3.6$  in Example 1 is an example of a linear function, with  $C_0 = 3.6$  and  $C_1 = 0$ . The function  $f(\xi) = 3\xi$  in Examples 2 and 3 is linear, with  $C_0 = 0$  and  $C_1 = 3$ . Now, what is the integral of the most general linear function  $C_0 + C_1x$ , over an arbitrary region of integration  $[a, b]$ ?

We'll prove that

$$\int_a^b (C_0 + C_1x) dx = C_0(b - a) + \frac{1}{2}C_1 (b^2 - a^2). \quad (13-24) \quad \boxed{\text{eq:genlin}}$$

After proving this result, we'll never again need to evaluate a Riemann sum for a linear function! We can just plug in the specific values of the constants

$C_0$ ,  $C_1$ ,  $a$  and  $b$  to calculate the integral.

Equation (13-24) is just a simple extension of the earlier results (13-8) and (13-21). The integral of a sum of functions is simply the sum of the integrals,

$$\int_a^b [f_1(x) + f_2(x)] dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx. \quad (13-25) \quad \boxed{\text{eq:sumof2}}$$

This is true because the Riemann sum of  $f_1 + f_2$  is the sum of  $f_1$  plus the sum of  $f_2$ , and, after taking the limit  $N \rightarrow \infty$ , the sums become the integrals in (13-25). Now note that the two terms on the right-hand side of (13-24) follow from the integrals in (13-8) and (13-21).

## 13.3 GRAPHICAL INTERPRETATION OF THE INTEGRAL

Graphical analysis of functions is an important technique in science and engineering. For example, consider the function

$$F(x) = x^2 e^{-x/5}, \quad (13-26) \quad \boxed{\text{eq:x2e}}$$

which might describe a cause-and-effect relationship between two variables,  $x$  and  $F$ , in a physical system. A scientist encountering this function, e.g., in a theoretical calculation, would immediately sketch its graph (Fig. [13-7](#)); the pictorial representation is easier to comprehend than the symbolic formula. The graph shows that as  $x$  increases from 0 to 30, the variable  $F$  starts at 0, then rises until  $F$  reaches a maximum at  $x = 10$  and then decreases more gradually to 0.

The derivative  $dF/dx$ , or  $F'(x)$ , is an important quantity in the analysis of a function  $F(x)$ . It is the *rate of change* of the variable  $F$  with respect to a change in  $x$ . The derivative function has a simple graphical interpretation, which we studied in Sec. 4.2: The derivative  $F'(x)$  is the slope of the graph of  $F(x)$  at  $x$ . For example, looking at Fig. [13-7](#) for the function in [\(13-26\)](#) we can immediately see that the slope starts at 0 at  $x = 0$ , becomes positive as  $x$  increases, but returns to 0 at the maximum ( $x = 10$ ); as  $x$  increases beyond this point the slope becomes negative but approaches 0 (from below) as  $x$  tends to  $\infty$ . All of the detailed properties of  $F(x)$  and  $F'(x)$  can be seen from the graph.

The slope of a graph is a “geometric” concept, because it involves the *shape* of the curve. The physical application of the function  $F(x)$  may be completely unrelated to geometry. But we introduce this geometric concept—the slope of the graph—as an abstract mathematical construction that will help us to analyze the function.

The definite integral  $\int_a^b F(x)dx$  is another important quantity in the analysis of a function. It is the *amount of a quantity* whose “density” (amount per unit of  $x$ ) is  $F(x)$ . This concept has many physical applications, as illustrated in Chap. 12. Can the graph of  $F(x)$  help us to analyze the integral?

The integral does have a simple graphical interpretation.  $\int_a^b F(x)dx$  is a certain *area* in the graph of  $F(x)$ : It is the area bounded above by the curve  $F(x)$ , bounded below by the  $x$  axis, and bounded on the sides by the vertical lines  $x = a$  and  $x = b$ . For example, the integral of the specific function  $F(x)$  in [\(13-26\)](#), from  $x = 5$  to  $x = 20$ , is the area of the shaded region in Fig. [13-7](#). We’ll prove that  $\int_a^b F(x)dx$  is equal to the shaded area presently.

It is important to understand that the “area of the graph” does not generally refer to a real physical area. In most applications,  $\int F(x)dx$  is not a physical area, or any other geometric quantity. It could be a mass, or kinetic

energy, or an electric field, etc. (cf. the examples in Chap. 12). We introduce the “area of the graph” as an abstract mathematical construction that helps us to analyze the integral.

Some applications of integration are in fact calculations of geometrical quantities, such as areas or volumes. For example, we considered area calculations in Sec. 12.1 as elementary examples of integration; see also, Chapter 16. In an area calculation the graphical interpretation of the integral is identical to the actual application. But in other applications the graphical interpretation is just an aid to understanding. Then the actual meaning of the integral is not a *physical* area but some other physical quantity—mass, kinetic energy, electric field, etc.

The following statements summarize the *meaning* and *graphical interpretation* of the definite integral.

**Meaning:** The integral  $\int_a^b g(x)dx$  is the total amount of a quantity whose density is  $g(x)$ , for  $x$  between  $a$  and  $b$ .

**Graphical interpretation:** The integral  $\int_a^b g(x)dx$  is the area bounded above by  $g(x)$  and below by the  $x$  axis, between  $x = a$  and  $x = b$ , on a graph of  $g(x)$ .

### 13.3.1 Proof of the graphical interpretation

Figure [fig:proofGI](#) illustrates the graph of  $g(x)$ . Subdivide the range of integration  $[a, b]$  into  $N$  small segments of width  $\Delta x$ . For each segment two rectangular strips are shown in the figure, which have width  $\Delta x$  and height  $g_{\max}$  or  $g_{\min}$ . The area  $A$  under the curve may be bounded above and below.  $A$  is less than the sum of the strips if we take the maximum value of  $g(x)$  in the  $k$ th strip as the height of the strip;  $A$  is greater than the sum of the strips if we take the minimum value of  $g(x)$  in the  $k$ th strip as the height of the strip. The areas of these larger and smaller strips (= width times maximum or minimum height, respectively) are  $g_k^{\max}\Delta x$  and  $g_k^{\min}\Delta x$ . Adding all the strip areas, the bounds on the area  $A$  under the curve are

$$\sum_{k=1}^N g_k^{\min}\Delta x \leq A \leq \sum_{k=1}^N g_k^{\max}\Delta x. \quad (13-27) \quad \boxed{\text{eq:bds}}$$

Now take the limit  $\Delta x \rightarrow 0$  and  $N \rightarrow \infty$ , with  $N\Delta x = b - a$ , a constant. The sums in [\(13-27\)](#) both tend to the same limit, namely the integral, by the definition [\(13-4\)](#). Either sum serves as the Riemann sum for the integral. The two bounds squeeze together to the value of the integral as  $\Delta x \rightarrow 0$ .

Therefore the area  $A$  under  $g(x)$  must be equal to the integral,

$$A = \int_a^b g(x)dx. \quad (13-28) \quad \boxed{\text{eq:A=int}}$$

Figure [13-28](#) gives an intuitive picture of the graphical interpretation of the definite integral. Integration is equivalent to subdividing the integration region into small elements, adding the elemental contributions (either  $g_k^{\max}\Delta x$  or  $g_k^{\min}\Delta x$ ) and taking the limit  $\Delta x \rightarrow dx$ . The result is the area under the curve because  $g(x)dx$  is the elemental area (height times width) of an infinitesimal element.

### 13.3.2 Negative integrals

In the discussion leading to [\(13-28\)](#) it was tacitly assumed that  $g(x)$  is positive in the integration region  $[a, b]$ . What about a function  $f(x)$  that is negative for some part of the region, as illustrated in [Fig. 13-29](#)? The elemental contribution  $f(x_k)\Delta x$  to the Riemann sum is negative for a segment in the  $x$  region  $[c, d]$  where  $f(x)$  is negative. If we interpret the integral as an area, what is the meaning of *negative* area? In the graphical interpretation we must understand negative area as area *below* the  $x$  axis. For example, in [Fig. 13-29](#), the integral of  $f(x)$  from  $x = a$  to  $x = b$  has three parts: positive area for  $x$  from  $a$  to  $c$ , where the curve is above the  $x$  axis; negative area for  $x$  from  $c$  to  $d$ , where the curve is below the  $x$  axis; and positive area for  $x$  from  $d$  to  $b$ . The integral from  $a$  to  $b$  is the sum of these three areas.

So the basic idea is that  $\int f(x)dx$  is the area between the curve of  $f(x)$  and the  $x$  axis. Where  $f(x)$  is positive, the curve is above the  $x$  axis and the area is positive; where  $f(x)$  is negative, the curve is below the  $x$  axis and the area is negative. In the graphical interpretation of the integral ( $\int f(x)dx = \text{area}$ ) regions where the curve is below the  $x$  axis count as negative area.

**Example 4.** Consider the sine function. [Figure 13-30](#) is a graph of  $\sin x$ , for  $x$  between 0 and  $2\pi$ . What is the integral from  $x = 0$  to  $x = 2\pi$ ?

**Solution.** The integral from 0 to  $\pi$  is *positive*, and in fact equal to  $+2$ .<sup>4</sup> The value  $+2$  is the area of the region bounded by the first half-cycle of the sine curve and the  $x$  axis; this area is *positive* because the curve lies *above* the  $x$  axis. The integral from  $\pi$  to  $2\pi$  is negative, and equal to  $-2$ . The value  $-2$  is the area bounded by the second half-cycle of  $\sin x$  and the  $x$  axis, and it is *negative* because the curve lies *below* the  $x$  axis. The integral from 0 to  $2\pi$

<sup>4</sup>We'll learn to evaluate this integral in Chapter 15.

is 0 because positive and negative contributions cancel,

$$\int_0^{2\pi} \sin x \, dx = 0. \quad (13-29)$$

## 13.4 DISTANCE TRAVELED IN ACCELERATING MOTION

Consider an object M moving along a line<sup>5</sup> with a varying velocity  $v(t)$ . The acceleration is

$$a(t) = \frac{dv}{dt}. \quad (13-30)$$

In a graph of velocity  $v$  versus time  $t$ , the slope of the curve at  $t$  is the acceleration at that time. What is the distance traveled by the object during some time interval, say from time  $t_1$  to  $t_2$ ?

The distance is the integral of the velocity,

$$D = \int_{t_1}^{t_2} v(t) dt. \quad (13-31) \quad \boxed{\text{eq:intv}}$$

The reason is because  $v$  is the distance per unit time, so that  $v dt$  is the distance traveled during the elemental time interval  $dt$ . Adding all the elemental distances, i.e., integrating the velocity, gives the total distance traveled. We could make the explanation more rigorous by describing a Riemann sum: Subdivide the interval  $[t_1, t_2]$  into small elements  $\Delta t$ , and write

$$D \approx \sum_{k=1}^N v(t_k) \Delta t; \quad (13-32)$$

the sum approximates  $D$  because  $v(t_k) \Delta t$  approximates the distance traveled during  $\Delta t$ . In the limit  $\Delta t \rightarrow 0$ , the approximation becomes exact. But the limit of the Riemann sum is, by definition, just the integral  $\int_{t_1}^{t_2} v(t) dt$  (eq:intv).

**Positive or negative displacements**

Suppose the velocity  $v(t)$  is negative,  $v = dx/dt < 0$ . Negative velocity means that the object M is moving toward smaller values of the coordinate  $x$ . Then  $\int v dt$  is negative, and  $\int_{t_1}^{t_2} v dt$  (eq:intv) would give a negative distance. A better terminology is to call  $D$  the *displacement*, which is positive if the object moves to a larger coordinate, or negative if the object moves to a smaller coordinate. Then the *distance* is defined as the absolute value of  $D$  (which must be positive).<sup>6</sup>

**Example 5.** What is the distance traveled from time  $t = 0$  to time  $t = T$  if  $v$  is constant?

**Solution.** We know that  $\int_{t_1}^{t_2} C dt = C(t_2 - t_1)$  by the general result  $\int_{t_1}^{t_2} C dt = C(t_2 - t_1)$  (eq:intCa).

<sup>5</sup>We name the object M for ‘moving.’

<sup>6</sup>For motion in three dimensions, displacement is a vector  $\mathbf{D}$  and distance is the scalar  $|\mathbf{D}|$  (the length of the vector) which must be positive.

Thus, for constant velocity  $v_0$  from  $t_1 = 0$  to  $t_2 = T$ ,

$$D = v_0 T. \quad (13-33)$$

The result is just velocity  $\times$  time, familiar from grade school.

**Example 6.** What is the distance traveled from time  $t = 0$  to time  $t = T$  if there is constant acceleration?

**Solution.** The velocity, for constant acceleration  $a$ , is  $v(t) = v_0 + at$  where  $v_0$  is the initial velocity. Using the general formula (13-24), the displacement is

$$D = \int_0^T (v_0 + at) dt = v_0 T + \frac{1}{2} a T^2. \quad (13-34)$$

This formula was used in Chap. 9 for motion with constant acceleration.

**Example 7.** What is the distance traveled from time  $t = 0$  to time  $t = T$  if the velocity is  $v(t) = U \sin(\pi t/T)$ ? Figure 13-24 shows a graph of  $v$  versus  $t$ .

**Solution.** The distance traveled is the integral, i.e., the area under the curve in Fig. 13-24,

$$D = \int_0^T U \sin \frac{\pi t}{T} dt. \quad (13-35)$$

We will learn to calculate this integral in Chapter 15. The result is  $D = 2UT/\pi$ .

**Example 8.** Consider the velocity function  $v(t)$  whose graph is shown in Fig. 13-25. The object M is located at the origin at time  $t = 0$ . Is the coordinate positive or negative at  $t = t_{\text{FIN}}$ ?

**Solution.** The “area under the curve” from  $t = 0$  to  $t_{\text{FIN}}$  is clearly negative. This area is the integral  $\int v dt$ , i.e., the displacement. The displacement is negative, so the final coordinate is negative. The net change of position during the time interval from 0 to  $t_{\text{FIN}}$  is to the left (with positive displacement defined to be to the right).

\* \* \*

**Summary** The three basic functions of kinematics<sup>7</sup> are  $x(t)$ ,  $v(t)$ , and  $a(t)$ —position, velocity, and acceleration. Recall from Chapter 9 the differ-

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<sup>7</sup> Kinematics is the analysis of motion.

ential equations obeyed by these functions,

$$\frac{dx}{dt} = v(t) \quad \text{and} \quad \frac{dv}{dt} = a(t).$$

We may now add an *integral equation*,

$$x(t) = x_0 + \int_0^t v(t') dt',$$

because by <sup>leg: intv</sup>(13-31) the integral of  $v(t')$  from  $t' = 0$  to  $t' = t$  is the displacement during the time interval  $[0, t]$ . Similarly, relating  $v$  and  $a$ ,

$$v(t) = v_0 + \int_0^t a(t') dt'.$$

The connection between the differential equation and the corresponding integral equation (e.g.,  $v$  is  $dx/dt$  and  $x$  is  $\int v dt$ ) is an example of the *fundamental theorem of calculus*—the subject of Chapter 14.

## 13.5 SUMMARY

Two basic mathematical ideas are contained in this chapter.

- The definition of the integral is

$$\int_a^b f(x)dx = \lim_{\Delta x \rightarrow 0} \sum_{k=1}^N f(x_k)\Delta x \quad (13-36)$$

where  $\{x_0, x_1, x_2, \dots, x_N\}$  is the subdivision of  $[a, b]$  into  $N$  segments (elements) of length  $\Delta x$ . The right-hand side is called the Riemann sum.

- The integral  $\int_a^b f(x)dx$  is equal to *the area under the curve* in a graph of  $f(x)$ . More precisely it is the area of the region bounded by the curve, the  $x$  axis, and the lines  $x = a$  and  $x = b$ .

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