Pavel, I've been looking at your calculation of the $\overline{MS} \to DIS$ transformation. The question is, why is the transformation divergent as $\epsilon \to 0$? (ϵ is the regulator in the x integral.) I don't know how to do the calculation, but here are some comments about a <u>simpler</u> <u>related</u> <u>calculation</u> that may be helpful.

Consider the distribution

$$C_{++}(x) \equiv [(f)_+ \otimes (g)_+](x)$$

where $f(y) = \frac{1}{1-y}$ and $g(z) = \frac{1}{1-z}$.

Let's calculate

$$\Phi \equiv \int_0^1 C_{++}(x)\, \varphi(x)\, dx$$

where $\varphi(x)$ is some well-behaved function. Then

$$\Phi = \int_0^1 \frac{dy}{1-y} \int_0^1 \frac{dz}{1-z} \Big[\varphi(yz) - \varphi(y) - \varphi(z) + \varphi(1)\Big]$$

$$\equiv \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \Phi^{(4)}$$

We'll look at the 4 terms. However, following Pavel, we'll regulate the integrals by replacing the upper endpoints ($y=1$ and $z=1$) by $1-\epsilon$ where $\epsilon \to 0$. Then

$$\Phi^{(4)} = \int_0^{1-\epsilon} \frac{dy}{1-y} \int_0^{1-\epsilon} \frac{dz}{1-z}\, \varphi(1) = \varphi(1)(\ln\epsilon)^2;$$

also,

$$\Phi^{(2)} = \Phi^{(3)} = -\int_0^{1-\epsilon} \frac{dy}{1-y} \int_0^{1-\epsilon} \frac{dz}{1-z}\, \varphi(z)$$

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$$\Phi^{(2)} = \Phi^{(3)} = -\int_0^{1-\epsilon} \frac{dy}{1-y} \int_0^{1-\epsilon} \frac{dz}{1-z} \varphi(z)$$

$$= -(-\ln\epsilon)(-\varphi(1)\ln\epsilon + F_a + O(\epsilon))$$

$$= -\varphi(1)(\ln\epsilon)^2 + F_a \ln\epsilon + O(\epsilon \ln\epsilon)$$

where $F_a = \int_0^1 \frac{dz\,[\varphi(z) - \varphi(1)]}{1-z}$ (finite term);

finally,

$$\Phi^{(1)} = \int_0^{1-\epsilon} \frac{dy}{1-y} \int_0^{1-\epsilon} \frac{dz}{1-z} \varphi(yz)$$

$$= \varphi(1)(\ln\epsilon)^2 - 2F_a(\ln\epsilon) + F_b + O(\epsilon)$$

where F_b is finite as $\epsilon \to 0$. ($\Phi^{(1)}$ must have this form!) Then

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \Phi^{(4)} = F_b \quad \text{(finite)}$$

<u>Examples</u> These are "moments" of $C_{++}(x)$.

<u>Example 1</u> Consider $\varphi(\xi) = 1$. By direct calculation,

$$\Phi^{(4)} = (\ln\epsilon)^2$$

$$\Phi^{(2)} = \Phi^{(3)} = -(\ln\epsilon)^2 \quad (\text{i.e., } F_a = 0)$$

$$\Phi^{(1)} = (\ln\epsilon)^2 \quad (\text{i.e., } F_b = 0)$$

$\Phi = 0$. The 0^{th} moment is 0.

<u>Example 2</u> Consider $\varphi(\xi) = \xi$. By direct calc.

$$\Phi^{(4)} = (\ln\epsilon)^2$$

$$\Phi^{(2)} = \Phi^{(3)} = -(\ln\epsilon)^2 - \ln\epsilon \quad (\text{i.e., } F_a = -1)$$

$$\Phi^{(1)} = \left[-\ln\epsilon - 1\right]^2$$
$$= (\ln\epsilon)^2 + 2\ln\epsilon + 1 \qquad (\text{i.e., } F_b = 1)$$

$\Phi = 1$. The 1^{st} moment is 1.

<u>Example 3</u> $\varphi(\xi) = \xi^2$. In this case,

$$\Phi^{(4)} = (\ln\epsilon)^2$$
$$\Phi^{(2)} = \Phi^{(3)} = -(\ln\epsilon)^2 - \tfrac{3}{2}\ln\epsilon \qquad (\text{i.e., } F_a = -3/2)$$
$$\Phi^{(1)} = (\ln\epsilon)^2 + 3\ln\epsilon + 9/4 \qquad (\text{i.e., } F_b = 9/4)$$
$$\Phi = 9/4. \qquad (\text{the second moment})$$

<u>Example 4</u> $\varphi(\xi) = \xi^n$

$$\Phi^{(4)} = (\ln\epsilon)^2$$
$$\Phi^{(2)} = \Phi^{(3)} = -(\ln\epsilon)^2 + F\ln\epsilon \quad \text{where } F = \int_0^1 \frac{z^n - 1}{1-z}\,dz$$
$$\Phi^{(1)} = (\ln\epsilon)^2 - 2F\ln\epsilon + F^2$$
$$\Phi = F^2$$
$$F = -\int_0^1 \sum_{k=0}^{n-1} z^k\,dk = -\sum_{k=0}^{n-1} \frac{1}{k+1} = -\sum_{k=1}^{n} \frac{1}{k}.$$

<u>The distribution</u>

Now let's try letting $\varphi(\xi) = \delta(\xi - x)$.

What is $\Phi(x) \equiv \int_0^1 C_{++}(\xi)\,\varphi(\xi)\,d\xi$?

(So, $\Phi(x) = C_{++}(x)$ at least formally.)

What happens if we calculate $\Phi(x)$ from $\Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \Phi^{(4)}$?

$$\Phi^{(4)} = \int_0^{1-\epsilon}\frac{dy}{1-y}\int_0^{1-\epsilon}\frac{dz}{1-z}\,\delta(1-x)$$
$$= (\ln\epsilon)^2\,\delta(1-x);$$

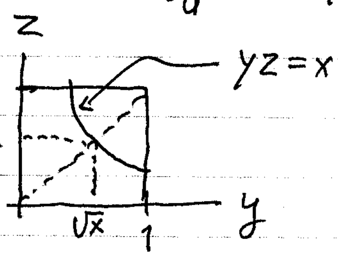
also,

$$\Phi^{(2)} = \Phi^{(3)} = -\underbrace{\int_0^{1-\epsilon}\frac{dy}{1-y}}_{-\ln\epsilon}\;\underbrace{\int_0^{1-\epsilon}\frac{dz}{1-z}\,\delta(z-x)}_{\frac{1}{1-x}\,\theta(1-\epsilon-x)}$$

$$= \ln\epsilon\;\frac{\theta(1-\epsilon-x)}{1-x}; \quad \leftarrow \boxed{\text{How do we take the limit } \epsilon\to 0\text{ ?}}$$

finally,

$$\Phi^{(1)} = \int_0^{1-\epsilon}\frac{dy}{1-y}\int_0^{1-\epsilon}\frac{dz}{1-z}\,\delta(yz-x)$$



$$\Phi^{(1)} = 2\int_{\sqrt{x}}^{1-\epsilon}\frac{dy}{1-y}\cdot\frac{1}{y}\,\frac{1}{1-x/y}$$
$$= 2\int_{\sqrt{x}}^{1-\epsilon}\frac{dy}{(1-y)(y-x)}$$

$$\Phi^{(1)} = \frac{-2\ln\epsilon}{1-x} + \frac{2}{1-x}\ln\left(\frac{1-x}{\sqrt{x}}\right) + O(\epsilon).$$

Now, can we combine $\Phi^{(1)} \dots \Phi^{(4)}$ and get a finite distribution $\Phi(x)$ $(= C_{++}(x))$.

The cancellations of $(\ln\epsilon)^2$ and $(\ln\epsilon)$ are not clear to me. And yet I know that every <u>moment</u> of $C_{++}(x)$ is finite as $\epsilon\to 0$!

I think the above calculation uses the same method that Pavel used: regulate the integrals and let $\epsilon\to 0$ at the end. If we can figure out this simpler problem, then we could do the more difficult case where $g(z) = \frac{\ln(1-z)}{1-z}$.