Convolution of plus distributions

Let $h(z)$ be the distribution

$$h(z) = [f_+ \otimes g_+](z) = \int_z^1 \frac{dx}{x} f_+\left(\frac{z}{x}\right) g_+(x). \tag{1}$$

The equation we need from Carl is

$$h(z) = \widetilde{h}_+(z) + A\delta(1-z), \tag{2}$$

where A is a constant (to be determined by comparing moments) and $\widetilde{h}_+(z)$ is the *plus-distribution* corresponding to a function $\widetilde{h}(z)$ given by

$$\begin{aligned}
\widetilde{h}(z) &= \int_{z/a}^1 \frac{dx}{x}\left[g\left(\frac{z}{x}\right) - xg(z)\right] f(x) \\
&+ \int_a^1 \frac{dx}{x}\left[f\left(\frac{z}{x}\right) - xf(z)\right] g(x) \\
&- g(z)F(z/a) - f(z)G(a).
\end{aligned} \tag{3}$$

(There were some misprints in Carl's notes.) Here

$$F(\xi) = \int_0^\xi f(x)dx \qquad \text{and} \qquad G(\xi) = \int_0^\xi g(x)dx. \tag{4}$$

The parameter a is in the range $z < a < 1$, but all the a dependence must cancel.

Example 1

Let $f(x) = \dfrac{1}{1-x}$ and $g(x) = \dfrac{1}{1-x}$.

This example is easy. The *function* $\widetilde{h}(z)$ is

$$\widetilde{h}(z) = \frac{1}{1-z} \ln\left[\frac{(1-z)^2}{z}\right]. \tag{5}$$

Then the *distribution* $h(z)$ is

$$h(z) = \left\{\frac{1}{1-z} \ln\left[\frac{(1-z)^2}{z}\right]\right\}_+ \quad . \tag{6}$$

Note that the constant A in (2) is 0, which, I guess, is always true. One can verify that the moments are correct, using Mathematica.

Example 2

Let $f(x) = \dfrac{1}{1-x}$ and $g(x) = \dfrac{\ln(1-x)}{1-x}$.

This example is harder. I can't do it without Mathematica. The answer is a long formula with PolyLogs.

I more or less have this example finished, but the final result is a long formula. I did check that the result for $\widetilde{h}(z)$ is independent of a. Then $A = 0$ in (2), and $h(z) = \widetilde{h}_+(z)$. I have not been able to verify that the moments are correct, because the integrals are hard even for Mathematica.

Dan