9pt

2. The Limit Concept

June 10, 2002

2.1 INTRODUCTION

There are three primary concepts in calculus—limit, derivative, and integral. The most important for applications in science and engineering are the derivative (which describes a rate of change) and the integral (which describes the total of many small parts). But the most basic of the three concepts is the limit, because the derivative and integral are defined as certain limits. So this chapter begins our study of calculus with the concept of the limit.

The limit of a function $f(x)$ at a point x_0 is the number that we obtain by evaluating $f(x)$ for values of x closer and closer to x_0 but not actually equal to x_0. We'll make this definition more precise later. The limit of $f(x)$ at x_0 is denoted by

$$\lim_{x \to x_0} f(x). \tag{2-1}$$ eq:notlim

The notation $x \to x_0$ is read as "x approaches x_0."

The quantity represented by (2-1) eq:notlim is a single number, if the limit exists. For many simple functions the limit is just the function value $f(x_0)$. But the limit is not necessarily equal to $f(x_0)$. For example, x_0 might be excluded from the domain of $f(x)$, so that $f(x_0)$ is not even defined; but the limit of $f(x)$ at x_0 may nevertheless be a well-defined number. Another example is when the function $f(x)$ has a discontinuity at x_0; then a limit at x_0 does not exist. In the examples below we'll analyze these and other cases, and learn when the limit exists as a well-defined number and when it does not exist.

The key idea to keep in mind is that the limit of $f(x)$ at $x = x_0$ is the number we obtain by evaluating $f(x)$ for x arbitrarily close to, but not actually equal to, x_0.

Comment on jargon

The meaning of the word "limit" in calculus is rather different from the everyday use of the word. In everyday usage, "limit" means some kind of *boundary* beyond which one cannot (or should not) go. The *speed limit* is the maximum allowed speed of a vehicle. To exceed this limit is to break the law.

But in calculus, a limit is not really a boundary. Think of the real numbers as a line. To find the limit of $f(x)$ at x_0 we might evaluate the function for points x close to x_0. But x_0 is not necessarily any kind of boundary—it may just be a point inside the domain. Then we *must* consider points both to the right and left of x_0.

In studying calculus and other technical subjects (including physics and engineering) we must use words according to their technical definitions, and ignore their meanings from everyday life. This kind of special terminology is called *jargon*. The meaning of a word in everyday usage can be fuzzy and ambiguous, but jargon is sharp and precise.

2.2 EXAMPLES OF LIMITS

In Sec. 2.3 we'll define the limit formally. But first let's examine some examples, to see what the issues are.

Example 1. What is the limit of the function $f(x) = (x+1)^2$ at $x = 2$?

Solution. The value of $f(x)$ at $x = 2$ is 9. If we evaluate $f(x)$ for numbers near 2 we obtain values near 9. For example,

$$f(2.01) = 9.0601 \quad \text{and} \quad f(1.99) = 8.9401. \tag{2-2}$$

The closer we push x toward 2, the closer $f(x)$ moves toward 9. So the limit of $f(x)$ at $x = 2$ is 9, and we write

$$\lim_{x \to 2} (x+1)^2 = 9. \tag{2-3}$$

A very good way to analyze limits is to look at the graph of the function in a neighborhood of the limit point x_0. Figure 2.1 shows $(x+1)^2$ versus x. The curve is smooth, and we see that as x moves to 2, from either the left or the right, the function value approaches 9.

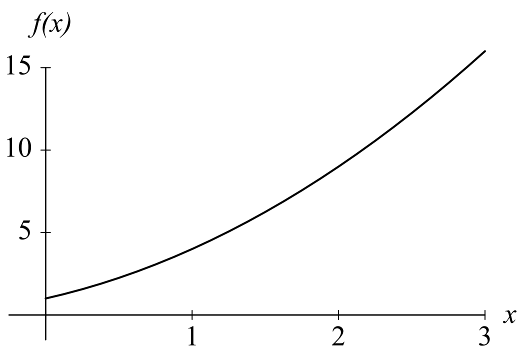

Figure 2.1: Example 1. The function $f(x) = (x+1)^2$ versus x. The limit at $x = 2$ is 9. fig:Ex1

Generalization. If the function $f(x)$ has a smooth graph and the point x_0 is in the domain of $f(x)$ then the limit of $f(x)$ at x_0 is simply the function value $f(x_0)$.

Not all functions have smooth graphs. The next two examples illustrate how the limit may differ from the function value.

Example 2. Consider the function $g(x)$ defined by

$$g(x) = \begin{cases} 2 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0. \end{cases} \tag{2-4}$$

A graph of $g(x)$ is illustrated in Fig. 2.2. What is the limit of $g(x)$ at $x = 0$?

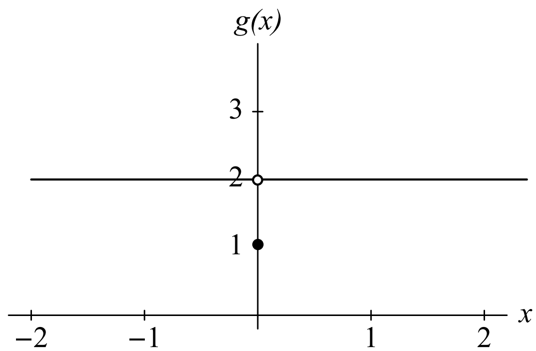

Figure 2.2: Example 2. The limit of $g(x)$ at $x = 0$ is 2, although the function value is $g(0) = 1$. fig:Ex2

Solution. If we evaluate $g(x)$ for points close to $x = 0$ we obtain 2, no matter how close we get to $x = 0$, as long as x is not exactly 0. Therefore,

$$\lim_{x \to 0} g(x) = 2. \tag{2-5}$$

But the function value at $x = 0$ is $g(0) = 1$. The limit at $x = 0$ exists and is well-defined: it is 2. But the limit is not equal to the function value.

To a mathematician, the function $g(x)$ in Example 2 is a perfectly good function. It satisfies the requirements for the definition of a function. To a physicist or engineer, the function $g(x)$ seems a bit artificial, i.e., unnatural. Is there any quantity in nature described by such a function—equal to 2 for all values of the independent variable except 0, and 1 for the value 0? This is not a function we are likely to encounter in natural science! But mathematics must allow all possibilities, and the example shows that a limit may exist and differ from the function value.

Example 3. Consider the function $h(x)$ defined by

$$h(x) = \begin{cases} 2 & \text{for} \quad x \geq 0 \\ 1 & \text{for} \quad x < 0. \end{cases} \tag{2-6}$$

A graph of $h(x)$ is shown in Fig. 2.3. In theoretical physics $h(x)$ is called a *step function*. It is the mathematically simplest example of a *discontinuity*. What is the limit of $h(x)$ at $x = 0$?

Solution. If we evaluate $h(x)$ for points near $x = 0$ we may get 1 or 2. If x is negative we get $h(x) = 1$; if x is positive we get $h(x) = 2$. The limit

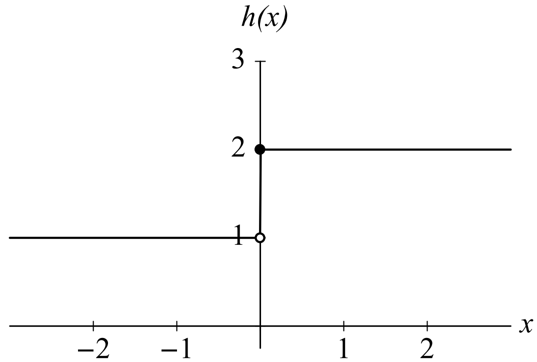

Figure 2.3: Example 3. The function $h(x)$ has a discontinuity at $x = 0$.

fig:Ex3

at $x = 0$ is undefined because the function doesn't have a unique value for points approaching $x = 0$. One could say that there are *two limits*: $h(x)$ approaches 2 as x approaches 0 from the positive side; and $h(x)$ approaches 1 as x approaches 0 from the negative side. (These two values are called *one-sided limits*.) But there is no single limit, i.e.,

$$\lim_{x \to 0} h(x) \ \text{ does not exist.} \tag{2-7}$$

We'll see in the formal definition of the limit (next section) that *all points* sufficiently near x_0 must have function values near the limit in order for the limit to be defined. Because $h(x)$ has a discontinuity at $x = 0$ the condition cannot be satisfied; h at points to the right of 0 is quite different from h at points to the left.

Discontinuities occur in nature, or at least in our theories of nature. In fact they are common. For example, at the surface of a lake there is a discontinuity of mass density: Just below the surface the density is $1\,\text{g/cm}^3$ (water) while just above it is $10^{-3}\,\text{g/cm}^3$ (air). On the atomic scale the discontinuity is different, but we would not use atomic physics to describe water waves on the surface of the lake. So a practical theory of hydrodynamics must allow *discontinuity* of density. Or, as another example, the electric field is discontinuous at the surface of a charged metal object: The electric field is 0 inside a conductor and σ/ϵ_0 just outside the conductor where σ is the charge per unit area on the surface. On the atomic scale the discontinuity is different, but we would not use atomic physics to calculate the electric field of a van de Graaff generator. So a practical theory of electrostatics must allow *discontinuity* of the field. These physical examples show that discontinuous functions, like $h(x)$ in Example 3, must be included in our mathematics.

Examples with zero over zero

Example 4. Let

$$F(x) = \frac{1-x}{1-x^2}. \qquad (2\text{-}8)$$

eq:Ex4

What is the limit of $F(x)$ at $x = 1$?

Solution. The point $x = 1$ is not in the domain of $F(x)$, because the denominator in (2-8) is 0 at $x = 1$. *Division by* 0 *is undefined!* So this example asks for the limit of a function at a point that is not in the domain.

If we naively try to evaluate $F(x)$ at $x = 1$, we obtain $0/0$. This "object" is not a number. It is an indeterminate result, because division by 0 is just *undefined* in mathematics. However, the *limit* at $x = 1$ may still exist.

Figure 2.4 shows a graph of $F(x)$ versus x. It is clear from the graph that the limit of $F(x)$ at $x = 1$ is well-defined and is a number near 0.5. In fact we'll prove that the limit is exactly 1/2.

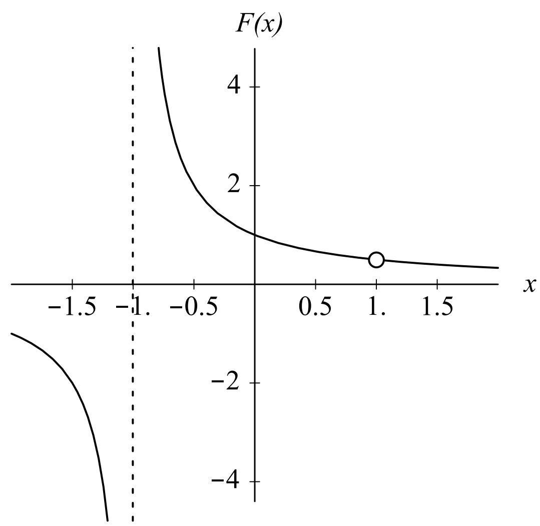

Figure 2.4: Example 4. The function $F(x)$ is undefined at $x = 1$, but has a well-defined limit; the limit at $x = 1$ is 1/2. Example 7. The function is singular, and has no limit, at $x = -1$.

fig:Ex4

If $x \neq 1$ then we may simplify the expression for $F(x)$ by noting that $1-x^2 = (1-x)(1+x)$ and canceling the common factors $(1-x)$ in the numerator and denominator,

$$\frac{1-x}{1-x^2} = \frac{1}{1+x} \qquad \text{provided} \qquad x \neq 1. \qquad (2\text{-}9)$$

Now, if we evaluate this simplified form for x near 1 we obtain a number near 1/2; and the closer x approaches 1 the closer the value of F approaches 1/2. So the limit of $F(x)$ is

$$\lim_{x\to 1}\frac{1-x}{1-x^2}=\frac{1}{2}. \tag{2-10}$$

This example is a case where the limit exists at a point where the function is undefined.

Generalization. If the naive evaluation of a function $f(x)$ at $x=x_0$ gives the indeterminate result 0/0, then *we have more work to do to determine the limit of $f(x)$ at x_0*. Finding 0/0 does not mean that the limit does not exist: The previous example has 0/0 for the naive evaluation, but the limit does exist. However, to find the limit we must analyze the function more carefully than just the naive evaluation.[1]

Example 5. Find the limit at $x=3$ of

$$G(x)=\frac{x^3-27}{x-3}. \tag{2-11}$$ eq:examp5

Solution. The naive evaluation gives 0/0. We have more work to do to determine the limit as $x\to 3$. We should try to cancel the factor $x-3$ that is making the numerator and denominator 0. Note that we can factor the numerator as

$$x^3-27=(x-3)(x^2+3x+9). \tag{2-12}$$ eq:factd

Therefore, for $x\neq 3$ the factor $(x-3)$ in (2-12) cancels the denominator in (2-11) and the function is

$$G(x)=x^2+3x+9. \quad (\text{for } x\neq 3) \tag{2-13}$$

As x approaches 3 the function approaches 27, so

$$\lim_{x\to 3}\frac{x^3-27}{x-3}=27. \tag{2-14}$$

Example 6. Consider the function

$$A(\theta)=\frac{\sin\theta}{\theta}. \tag{2-15}$$

(Whenever the sine or cosine function appears in calculus, it is understood that the argument (here, θ) is expressed in radians.) What is the limit of $A(\theta)$ at $\theta=0$?

Solution. This is another case where naive evaluation gives 0/0. As in the

[1] L'Hôpital's Rule is a standard technique, described in Chap. 7.

other cases, 0/0 tells us nothing. We have more work to do. But unlike the other cases, which could be simplified algebraically, there is no simple way to factor out θ from $\sin\theta$; the function $\sin\theta$ is not algebraic. So how can we find the limit at $\theta = 0$?

We could plug in some specific values of θ to get an idea of the numerical values of $\sin\theta/\theta$. Please use a calculator to verify the values in this table:

θ	$A(\theta)$
1	0.841
0.1	0.9983
0.01	0.999983
0.001	0.99999983

Plainly, the limit is 1. But this is just a numerical experiment, not a proof. We could also plot a graph of $A(\theta)$, Figure 2.5.[2] Again it is clear that the limit at $\theta = 0$ is 1, but this is still a numerical experiment.

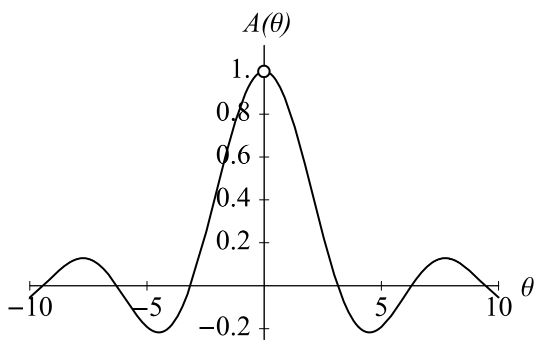

Figure 2.5: Example 6. The function $A(\theta)$ is undefined at $\theta = 0$; the limit at $\theta = 0$ is 1. fig:sinex

To prove rigorously that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \tag{2-16}$$

requires methods of calculus. The slope of $\sin\theta$ at $\theta = 0$ is 1 (by calculus) so if θ is very small, a valid approximation of the function is

$$\sin\theta \approx \theta \quad \text{for small} \quad \theta. \tag{2-17}$$

Then the function $\sin\theta/\theta$ is 1 in the same approximation. Figure 2.6 shows a graph of $\sin\theta$ and its linear approximation θ, versus θ. The figure illustrates that the line with slope 1 is the tangent line at $\theta = 0$, so that $\sin\theta \approx \theta$. Thus the limit of $A(\theta)$ as $\theta \to 0$ is 1. We'll return to this example in Chapter 11 when we analyze the slope (or, derivative) of the sine function.

[2] Please reproduce this graph using a graphing calculator.

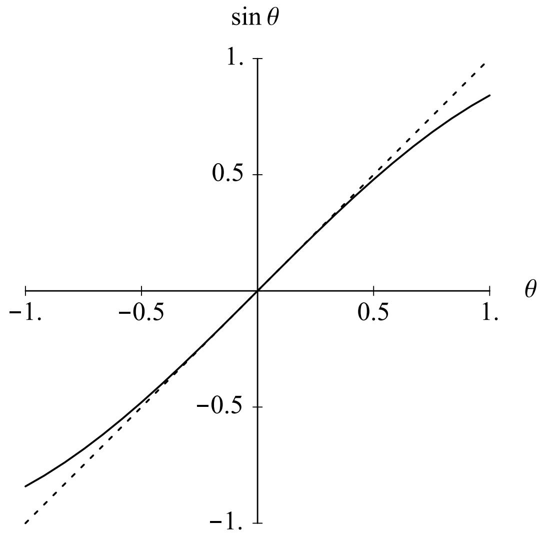

Figure 2.6: Comparison of $\sin\theta$ (solid curve) and θ (dashed curve). For small θ, $\sin\theta$ is approximately equal to θ. fig:sinandlin

Example 7. Consider again the function $F(x)$ in Example 4. What is the limit of $F(x)$ at $x = -1$?

Solution. The naive evaluation is $2/0$, which is undefined. Figure fig:Ex4 2.4 shows a graph of $F(x)$. As x approaches -1 from the right the function increases without bound to $+\infty$; as x approaches -1 from the left the function is negative and decreases without bound to $-\infty$. Obviously there is no limit—no number that the function is near for all points in a small neighborhood of $x = -1$. The limit does not exist.

2.3 FORMAL DEFINITION OF THE LIMIT — EPSILONS AND DELTAS

Now that we have seen some examples, we are ready to study a precise definition of the limit.

> ***Definition.*** The limit of $f(x)$ at x_0 is a number A if for every ϵ, no matter how small but with $\epsilon > 0$, there exists a δ (greater than 0) such that $|f(x) - A| < \epsilon$ for all x with $0 < |x - x_0| < \delta$.

Figure 2.7 illustrates the definition pictorially. No matter how small ϵ is, there exists a δ such that all the points in the *domain interval* $(x_0 - \delta, x_0 + \delta)$ (omitting x_0) have $f(x)$ in the *range interval* $(A - \epsilon, A + \epsilon)$. No matter how small the ϵ-neighborhood of A is chosen, there exists a δ-neighborhood of x_0 such that all points in it (omitting x_0) have $f(x)$ in the ϵ-neighborhood.

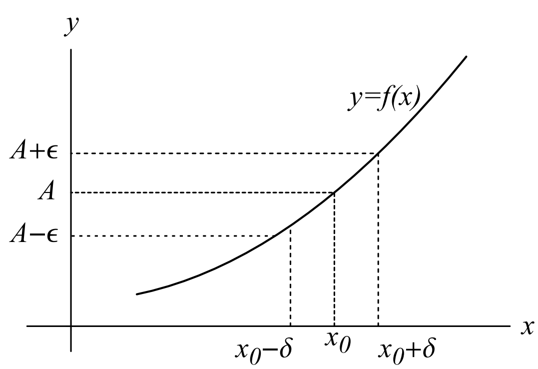

Figure 2.7: Illustration of the formal definition of the limit. For an arbitrarily small ϵ (but greater than 0) $|f(x) - A|$ is less than ϵ for all x in the interval $[x_0 - \delta, x_0 + \delta]$. Then the limit of $f(x)$ at $x = x_0$ is A. fig:epdel

To prove that some function $F(x)$ has a limit, say C, at $x = x_0$, we should in principle prove that all the points (omitting x_0) in some sufficiently small neighborhood of x_0 have $F(x)$ within the interval $(C - \epsilon, C + \epsilon)$ for any arbitrarily small ϵ. But that kind of formal proof can be rather tedious, so we usually rely on simpler methods, as in the examples in Sec. 2.2. However, if there is any question about the limit then the rigorous proof must be supplied.

2.4 GENERAL THEOREMS ON LIMITS

Suppose $f(x)$ approaches A, and $g(x)$ approaches B, as x approaches x_0,

$$\lim_{x\to x_0} f(x) = A \qquad \text{and} \qquad \lim_{x\to x_0} g(x) = B. \tag{2-18}$$

Then *combinations* of $f(x)$ and $g(x)$ have known limits. We'll prove

$$\lim_{x\to x_0} [f(x) + g(x)] = A + B, \tag{2-19}$$

eq:gen1

$$\lim_{x\to x_0} f(x)g(x) = AB, \tag{2-20}$$

eq:gen2

and several other related results.

Let ϵ be an arbitrarily small number (but $\epsilon > 0$). To prove (2-19) we must show that there exists a δ (greater than 0) such that

$$|f(x) + g(x) - A - B| < \epsilon \quad \text{for all } x \text{ with} \quad 0 < |x - x_0| < \delta.$$

Note that δ_1 and δ_2 exist such that

$$|f(x) - A| < \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \delta_1,$$
$$|g(x) - B| < \frac{\epsilon}{2} \quad \text{for} \quad |x - x_0| < \delta_2.$$

By the *triangle inequality*,

$$|f(x) + g(x) - A - B| \le |f(x) - A| + |g(x) - B|. \tag{2-21}$$

eq:prg1

Now let δ be the smaller of δ_1 and δ_2. Then for $|x - x_0| < \delta$,

$$|f(x) + g(x) - A - B| < \epsilon/2 + \epsilon/2 = \epsilon. \tag{2-22}$$

Hence (2-19) is proven.

The inequality (2-21) is an example of the *triangle inequality*. Let α and β be any real numbers. The triangle inequality is

$$|\alpha + \beta| \le |\alpha| + |\beta|. \tag{2-23}$$

eq:triangle

If α and β have the same sign, then (2-23) is true because $|\alpha+\beta| = |\alpha| + |\beta|$. If α and β have opposite signs, then (2-23) is true because $|\alpha+\beta| < |\alpha| + |\beta|$.

Next we'll prove (2-20). Again, let ϵ be arbitrarily small (but > 0). We must show that there exists a δ such that

$$|f(x)g(x) - AB| < \epsilon \quad \text{for all } x \text{ with} \quad 0 < |x - x_0| < \delta.$$

Note that δ_1 and δ_2 exist such that

$$|f(x) - A| < \epsilon_1 \quad \text{for} \quad |x - x_0| < \delta_1,$$
$$|g(x) - B| < \epsilon_2 \quad \text{for} \quad |x - x_0| < \delta_2,$$

for any ϵ_1 and ϵ_2. Start with the identity

$$fg - AB = (g - B)f + (f - A)g - (f - A)(g - B). \tag{2-24}$$

(For short we write f for $f(x)$ and g for $g(x)$.) Using the *triangle inequality*, and assuming for simplicity that A and B are positive,

$$\begin{aligned} |fg - AB| &\leq |(g-B)f| + |(f-A)g| + |(f-A)(g-B)| \\ &= |g-B||A+(f-A)| + |f-A||B+g-B| + |f-A||g-B| \\ &\leq |g-B|(A+|f-A|) + |f-A|(B+|g-B|) + |f-A||g-B| \\ &= A|g-B| + B|f-A| + 3|f-A||g-B| \\ &< A\epsilon_2 + B\epsilon_1 + 3\epsilon_1\epsilon_2. \end{aligned}$$

By making ϵ_1 and ϵ_2 small enough, we can certainly make the final expression be less than ϵ. Then, taking δ to be the smaller of δ_1 and δ_2, $|fg - AB| < \epsilon$ for $|x - x_0| < \delta$. The theorem is proven.

Some related general theorems, proofs of which are left as exercises, are

$$\lim_{x\to x_0} cf(x) = cA \quad (c \text{ a constant}), \tag{2-25}$$

$$\lim_{x\to x_0} [f(x) - g(x)] = A - B, \tag{2-26}$$

$$\lim_{x\to x_0} f(x)/g(x) = A/B, \tag{2-27}$$

$$\lim_{x\to x_0} [f(x)]^p = A^p. \tag{2-28}$$

The general theorems are useful, because they can be used to evaluate limits of functions that can be separated into two or more parts. For reference, the general theorems are recorded in Table 2.1.

Example 8. Determine the limit

$$\lim_{x\to 5} \frac{(x^2+4)\sqrt{x+5}}{2x}. \tag{2-29}$$

Solution. Treat the function as three factors: $(x^2+4) \to 29$; $(2x) \to 10$; and $\sqrt{x+5} \to \sqrt{10}$. By the general theorem (2-20), extended to 3 factors, the limit of the product is the product of the limits, $29/\sqrt{10}$.

$\lim(f+g) = \lim f + \lim g$
$\lim(f-g) = \lim f - \lim g$
$\lim(Cf) = C \lim f$
$\lim(fg) = \lim f \times \lim g$
$\lim(f/g) = \lim f / \lim g$
$\lim(f^p) = (\lim f)^p$

Table 2.1: General Theorems for Limits. It is assumed that $f(x)$ and $g(x)$ have well-defined limits at the limit point; then the limits of combinations of $f(x)$ and $g(x)$ are listed in the table. (C and p denote constants.) tbl:gt

2.5 INFINITY

What is infinity? It is not a number, but a limit. Infinity (∞) is the limit of a variable that increases without bound. So, for example, moving to the right on the real line at a constant pace is approaching infinity.

Infinity may seem like mere abstract mathematics, but infinity occurs often in calculations in science and engineering. Of course no real physical system can be truly infinite (except, perhaps, the entire cosmos). But physicists will often need to analyze the infinite limit in physical theories. For example, the energy flux *at infinity* is a useful concept in antenna theory. Or, a nuclear scattering experiment measures the subatomic particles *at infinity*. In these examples, "at infinity" means in the limit of distances that are much larger than any dimension of the primary physical system (antenna or nucleus, respectively).

In the study of functions, infinity may appear in two ways: either as $x \to \infty$ or as $f \to \infty$. We'll explore these cases in the next two examples.

Example 9. What is the limit of $K(\xi) = (\xi^2 + 1)/(\xi^2 + 2)$ as $\xi \to \infty$?

Solution. The limit is 1,

$$\lim_{\xi \to \infty} \frac{\xi^2 + 1}{\xi^2 + 2} = 1. \tag{2-30}$$

This result should be clear. If ξ is very large ($\xi \gg 1$) then $\xi^2 + 1$ and $\xi^2 + 2$ are both essentially ξ^2, so the ratio is 1. More rigorously, note that

$$\frac{\xi^2 + 1}{\xi^2 + 2} = 1 - \frac{1}{\xi^2 + 2},$$

and the second term obviously approaches 0 as $\xi \to \infty$.

⋆

The formal definition of a limit as $x \to \infty$ is different from the case $x \to x_0$. Instead of small ϵ and small δ, we have small ϵ and large D:

> ***Definition.*** The limit of $f(x)$ as $x \to \infty$ is A if for every small ϵ there exists a large D such that $|f(x) - A| < \epsilon$ for all x with $x > D$.

The independent variable may also go to $-\infty$. The condition for the limit to be A' as $x \to -\infty$ is $|f(x) - A'| < \epsilon$ for all x with $x < -D$.

⋆

Example 10. In electrostatics, the magnitude of the force between two positive charges, q_1 and q_2, is

$$F(r) = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \tag{2-31}$$ eq:Coulomb

where r is the distance between the charges. What is the limit of $F(r)$ as $r \to 0$?

Solution. The limit is ∞. This is a case where the independent variable is finite but the function goes to infinity.

This singularity at $r = 0$ is somewhat unphysical. We are assuming that the charges are *points* so that the distance between them can go to 0. In electrical engineering any real charged object would have a nonzero size, so the force equation (2-31) would not hold down to $r = 0$. However, in atomic physics the electron is treated as a point charge. The singularity at $r = 0$ is modified by quantum theory, but nevertheless has great importance in the theory, leading to *renormalization of quantum electrodynamics*.

2.6 FINAL REMARK

The idea of the limit of a function $f(x)$ at $x = x_0$ is to study $f(x)$ for x approaching x_0, but not exactly equal to x_0. The function may be undefined at x_0 but still have a well-defined limit. Conversely, the function may be defined at x_0 but not have a limit.

Let's consider a final example. Suppose F is some physical quantity that varies with time t, i.e., a function $F(t)$. Now consider

$$R(h) \equiv \frac{F(t+h) - F(t)}{h} \tag{2-32}$$

where h is a small time interval; and regard $R(h)$ as a function of h for some specified time t. What is the limit of $R(h)$ at $h = 0$? R is undefined at $h = 0$ because of division by 0. However, the *limit* as $h \to 0$ is a well-defined number. This limit,

$$\lim_{h \to 0} \frac{F(t+h) - F(t)}{h}, \tag{2-33}$$

is the *derivative* of $F(t)$. It represents the instantaneous rate of change of the quantity F. The next few chapters are all about derivatives.

EXERCISES

Section 2: Examples of limits

2-1. *Cases in which the limit equals the function value.* For each case, determine the limit, sketch a graph of the function, and indicate the limit point on the graph.

(a) $\lim_{x\to 5} 2x^2$

(b) $\lim_{y\to -1/2} \frac{1}{y+1}$

(c) $\lim_{\theta\to 3\pi/4} \tan\theta$

(d) $\lim_{t\to 2} 10^{-t}$

(e) $\lim_{x\to 1} \frac{1}{\sqrt{x^2+1}}$

(f) $\lim_{x\to a} \frac{1}{\sqrt{x^2+b^2}}$

2-2. *Cases in which the limit point is not in the domain of the function.* For each case, naive evaluation gives 0/0, so more work is needed to figure out the limit. Determine the limit, sketch a graph of the function, and indicate the limit point on the graph.

(a) $\lim_{x\to 5} \frac{x^2-25}{x-5}$

(b) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

(c) $\lim_{\xi\to -2} \frac{\xi+2}{\xi^2-4}$

(d) $\lim_{x\to 1} \frac{x-1}{2x-\sqrt{x^2+3}}$

Hint: Multiply and divide by $2x+\sqrt{x^2+3}$.

(e) $\lim_{x\to c} \frac{\sqrt{x+1-c}-1}{x-c}$ (c a constant).

Hint: Use the same trick as in (d).

(f) $\lim_{\phi\to \pi} \frac{\tan\phi}{\pi-\phi}$

(g) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$

2-3. For each case, determine whether the limit exists. If it exists, give the limit; if it does not exist, explain why not.

(a) $\lim_{x\to -1} \frac{1}{1-x^2}$

(b) $\lim_{x\to 0} \sqrt{x^4}$

(c) $\lim_{\phi\to 0} \cot\phi$

(d) $\lim_{\phi\to 0} \sin\phi\times\cot\phi$

Section 3: Formal definition of the limit

2-4. Each of the cases below has the form $\lim_{x\to x_0} f(x) = A$. For each case, determine the maximum δ such that $|f(x)-A| < \epsilon$ for all x with $|x-x_0| < \delta$.

(a) $\lim_{x\to 2} x^2 = 4.$

(b) $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}.$

(c) $\lim_{x \to 1} \sqrt{x^2+1} = \sqrt{2}$.

(d) $\lim_{x \to 0} \frac{2x+3}{x^2+4} = \frac{3}{4}$.

Section 4: Infinity

2-5. *Cases in which the independent variable approaches* $+\infty$ *or* $-\infty$*.* For each case, determine the limit and sketch a graph of the function showing how it approaches the limit.

(a) $\lim_{x \to \infty} \frac{ax^2+b}{cx^2+d}$ where a, b, c, d are constants.

(b) $\lim_{x \to -\infty} \frac{(x+2)(x+3)}{x^2+1}$

(c) $\lim_{x \to \pm\infty} \frac{x}{\sqrt{x^2+1}}$

(d) $\lim_{x \to \infty} \frac{3x^2}{(3x^2+1)^2}$.

(e) $\lim_{x \to \infty} \frac{x^2}{\sqrt{x^2+1}}$.

(f) $\lim_{\xi \to \infty} \frac{\ln \xi}{\xi}$

(g) $\lim_{\xi \to -\infty} \xi^2 e^{\xi}$

General Exercises

2-6. Consider the function $F(x) = x^x$. What is the domain such that $F(x)$ is real and well defined? Plot a graph of $F(x)$. What is the limit of $F(x)$ at $x = 0$? (This is a one-sided limit because $x < 0$ is outside the domain.) Estimate the minimum value of $F(x)$.

[This function is very peculiar to a physicist. The functions x^p and c^x are familiar (where p and c are constants) but x^x never appears in physical science! With calculus one can show that the minimum of x^x is $(1/e)^{(1/e)}$.]

2-7. What is the limit of the function $P(x) = \sin(1/x)$ at $x = 0$?

Hint: Use a graphing calculator or computer graphics to plot a graph of the function, as well as you can. This is a very pathological function! It is continuous and bounded (between -1 and 1) but never settles down to a limit as x approaches 0.

Contents