Dirac Equation for free spin-$\frac{1}{2}$ particle

$$(i\not{\partial} - m)\psi = 0$$

1. Schrödinger eq.

$$\frac{\underline{p}^2}{2m} + V = E$$

$$\underline{p} \to \frac{\hbar}{i}\underline{\nabla}, \quad E \to i\hbar\frac{\partial}{\partial t} \quad \left(\begin{array}{c}\text{Quantum}\\ \text{Prescription}\end{array}\right)$$

$$\Rightarrow \left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

2. Klein-Gordon eq.

$$E^2 - \underline{p}^2c^2 = m^2c^4$$

or,

$$p^\mu p_\mu - m^2c^2 = 0$$

wher

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

$$p_\mu = g_{\mu\nu}p^\mu = \left(\frac{E}{c}, -p_x, -p_y, -p_z\right)$$

$$\left(p^\mu p_\mu = \frac{E^2}{c^2} - \underline{p}^2 = m^2c^2\right)$$

quantum replacement:

$$p_\mu \to i\hbar \partial_\mu \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\right)$$

$$x^\mu = (ct, x, y, z)$$

$$\Rightarrow \left(-\hbar^2 \partial^\mu \partial_\mu - m^2 c^2\right) \psi = 0$$

Or,

$$\left(\frac{-1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) \psi = \left(\frac{mc}{\hbar}\right)^2 \psi$$

$$\left[\begin{array}{l} \text{where} \quad \partial^\mu \equiv \dfrac{\partial}{\partial x_\mu} \\ \quad p_\mu = \left(\dfrac{E}{c}, -\underline{p}\right) \to i\hbar\left(\dfrac{1}{c}\dfrac{\partial}{\partial t}, \underline{\nabla}\right) \\ x_\mu = g_{\mu\nu} x^\mu, \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \end{array}\right]$$

K-G eq: (1) relativistic

(2) for Spin-0 particles

3. Dirac's derivation

(1) Assume

$$\left(p^\mu p_\mu - m^2c^2\right) = \left(\beta^\kappa p_\kappa + mc\right)\left(\gamma^\lambda p_\lambda - mc\right)$$

where β^κ and γ^λ are 8 coefficients to be determined.

(2) The R.H.S $\Rightarrow$

$$\beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc\left(\beta^\kappa - \gamma^\kappa\right)p_\kappa - m^2c^2$$

$\Rightarrow$ Require $\beta^\kappa = \gamma^\kappa$

$$p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

$\Rightarrow$ Require $(\gamma^0)^2 = 1, \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \quad \text{for } \mu \neq \nu$$

Or,

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}}$$

anti-commutator

Minkowski metric

(3) The "gamma matrices" γ^μ are

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}_{4\times4}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}_{4\times4}$$

where σ^j $(j=1,2,3)$ are Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(4) Thus,

$$\left(p^\mu p_\mu - m^2c^2\right) = \left(\gamma^\kappa p_\kappa + mc\right)\left(\gamma^\kappa p_\kappa - mc\right) = 0$$

$\Rightarrow$ Dirac eq. $\left(\gamma^\mu p_\mu - mc\right)\psi = 0$

(5) Quantum Substitution $\quad p_\mu \to i\hbar\partial_\mu$

$$\Rightarrow \left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi = 0$$

Or, $\left(i\hbar\not{\partial} - mc\right)\psi = 0$

(6) Plane-wave solutions :

① $\psi(\underline{r},t) \sim e^{\frac{-i}{\hbar}(p\cdot x)}\, u(p)$, (Particle states)

where $p\cdot x \equiv p_\mu x^\mu = (Et - \underline{p}\cdot\underline{r})$

$$(\gamma^\mu p_\mu - mc)\, u(p) = 0$$

Or, $(\not{p} - mc)\, u(p) = 0$, $(\not{p} \equiv \gamma^\mu p_\mu)$

② $\psi(\underline{r},t) \sim e^{\frac{+i}{\hbar}(p\cdot x)}\, v(p)$, (Anti-particle states)

$(\gamma^\mu p_\mu + mc)\, v(p) = 0$

Or $(\not{p} + mc)\, v(p) = 0$.

③ In the rest-frame : (e^-)

$u^{(1)}(p) \sim \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Spin-up electron

$u^{(2)}(p) \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Spin-down electron

$$v^{(1)}(p) \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{Spin-up positron } (e^+)$$

$$v^{(2)}(p) \sim \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{Spin-down positron}$$

<u>Note</u>: the plane-wave solution for Schrödinger eq is

$$\psi(\underline{r}, t) = \psi(\underline{r})\, e^{\frac{-iEt}{\hbar}}$$

Also,

$$e^{\frac{-i}{\hbar}(p\cdot x)} = e^{\frac{-i}{\hbar}(Et - \underline{p}\cdot\underline{r})}$$

$$= e^{+\frac{i}{\hbar}(\underline{p}\cdot\underline{r})} \cdot e^{\frac{-iEt}{\hbar}}$$

($\Rightarrow$ positive energy states.)