Dirac Equation for Spin-$\frac{1}{2}$ charged particle in external E.M. field

1. For free particle:

$$(i\partial^\mu \gamma_\mu - m)\psi = 0$$

2. Use quantum replacement

electric charge

$$p^\mu \equiv i\partial^\mu \longrightarrow p^\mu - eA^\mu$$

with $p^\mu = \left(\frac{E}{c}, \underline{P}\right)$ and $A^\mu = (V, \underline{A})$

vector potential

$\Rightarrow$ non-relativistic limit

$$\left(\underline{P} \to \underline{P} - e\underline{A}\right)$$

Scalar potential

3. Thus, for electron,

$$(i\not{D} - m)\psi = 0$$

$$\not{D} \equiv \gamma_\mu D^\mu, \quad iD^\mu = i\partial^\mu - eA^\mu$$

[redacted]

Applying Dirac Eq. to H-atom

1. Potential $V(\underline{r}) = \frac{-e^2}{r}$

(The E.M. field is treated classically, though the electron field is treated quantum mechanically.)

2 Energy levels found by expanding Dirac Eq. in the non-relativistic limit (i.e. $v \ll c$)

(1) The leading term reproduce the result of Schröding eq.

$$E_n^{(0)} = \frac{-me^4}{2\hbar^2} = -\alpha^2 mc^2 \left(\frac{1}{2n^2}\right)$$

$$= \frac{-13.6\,(eV)}{n^2}, \quad (n = 1, 2, 3 \cdots)$$

with $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}$

(fine structure Constant)

Bohr radius: $a_0 \equiv \frac{\hbar^2}{me^2} = 0.53 \times 10^{-8}\ cm$

(2) Include the relativistic kinematic correction (rel.) due to

$$T_{rel.} = \frac{\underline{p}^2}{2m} - \frac{\underline{p}^4}{8m^3c^2} + \cdots$$

from expanding

$$\Rightarrow \Delta H_{rel} = \frac{-1}{8m^3c^2}\underline{p}^4$$

$$T_{rel.} = \sqrt{\underline{p}^2c^2 + m^2c^4} - mc^2$$

(3) Include Spin-Orbit interaction

$$\Delta H_{so} = \frac{e^2}{2m^2c^2r^3}(\underline{L}\cdot\underline{S})$$

The magnetic dipole moment of electron is (with $e<0$)

$$\underline{\mu} = \frac{e}{mc}\underline{S} \quad \text{with} \quad \underline{S} = \frac{\underline{\sigma}}{2} \leftarrow \text{Pauli matrix}$$

$$\equiv g_e \mu_B \underline{S}, \quad \text{with}$$

$$\boxed{\text{Bohr magneton} \quad \mu_B \equiv \frac{|e|}{2m}}$$

$$\equiv \mu_e \cdot \underline{\sigma}$$

Thus,

$$\mu_e = \frac{e}{2mc}$$

$\Rightarrow$ Landé g-factor $\boxed{g_e = 2}$

Thus, Dirac Eq. "knows" that electron has spin-$\frac{1}{2}$, and it predicts $g_e = 2$.

$\Rightarrow$ fine-structure splitting (fs) ΔE_{fs} depends only on j, not on l.

For example,

$$2S_{1/2} \ (n=2,\ l=0,\ j=\tfrac{1}{2})$$

and $$2P_{1/2} \ (n=2,\ l=1,\ j=\tfrac{1}{2})$$

share the same energy.

But, in 1947, Lamb & Rutherford experiment showed $E(2S_{1/2}) > E(2P_{1/2})$

$$\Rightarrow \mu_e = \frac{e}{2m}(1 + \underbrace{k_e}_{\text{anomalous magnetic moment}})$$

with $k_e = \frac{1}{2}(g_e - 2)$ (Dirac eq. predicts $k_e = 0$)

$= 1 + \frac{\alpha}{2\pi} + \cdots$ (Predicted by QED)

$\simeq 1.00116\ldots$

$$\left(\frac{g-2}{2}\right)_e^{QED} = (0.5)\frac{\alpha}{\pi} - 0.32848\left(\frac{\alpha}{\pi}\right)^2 + 1.19\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$
$$= (1159652.4 \pm 0.4)\times 10^{-9}$$

$$\left(\frac{g-2}{2}\right)_\mu^{QED} = (0.5)\frac{\alpha}{\pi} + 0.76578\left(\frac{\alpha}{\pi}\right)^2 + 24.45\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$
$$= (1165851.7 \pm 2.3)\times 10^{-9}$$

Note: The leading term $0.5\left(\frac{\alpha}{\pi}\right)$ is the same for e^- and μ^-, while the α^2 term differs in both magnitude and sign.

For $(g-2)_\mu$, it's important to include also strong-interaction correction from

$$\Delta a_\mu = \frac{m_\mu}{12\pi^3}\int \frac{\sigma(e^+e^- \to \text{hadrons})}{s}\,ds$$
$$\simeq 7\times 10^{-8}$$

In H-atom, the $n=2$ energy-level splittings: