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Decay of Ordered and Chaotic Systems

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We study the decay of classical systems with regular and chaotic dynamics by investigating the escape of a particle from a container with a small hole. For the case of ergodic motion, we find an exponential decay law, whereas the nonchaotic system decays according to a power law.

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In the last decade there has been much interest in the classifiction of motion as chaotic or nonchaotic using simple model Hamiltonians. An important question in the case of finite systems is how a system, once excited, decays. We are familiar with exponential decay laws for single quantum states and more generally when there are many external channels and the conditions for quantum chaos exist. On a purely classical level, the dependence of the decay laws on chaoticity has not been investigated, to our knowledge. One expects exponential decay for chaotic Hamiltonians, and perhaps nonexponential behavior for more ordered dynamics. We present here a study for a system similar to the one considered by Sinai[1] which confirms this expectation.

We consider point particles moving in a rectangular box bouncing elastically off the walls. This is an idealization of the physical situation of nucleons confined inside a nucleus, for example. At low excitation energies, the nucleons have weak residual interactions with each other, and their motion is governed mainly by a mean-field potential. The quantum-mechanical realization of this physical scenario results in the nuclear shell model[2] or, simpler, in the Fermi-gas model.[3] Other physical cases which could be idealized in the above way are charged ions or electrons confined in a Paul trap.[4] We allow our system to decay by providing a small window in one of the container walls through which particles are allowed to escape. This is physically similar to the wall and window formalism[5] which was employed to describe the influence of friction and nucleon exchange in nuclear reactions. The interactions between particles are idealized by placing a stationary circular scattering center inside our container; this leads to ergodic motion. Physical situations corresponding to this limit are nucleons at high excitation energy inside a nucleus or atoms in a highly excited complex molecule.

We first discuss the two-dimensional case. The example of regular motion is then a point particle on a rectangular billiard table. It moves under the influence of elastic reflections on piecewise linear trajectories. Every reflection off one of the walls can be represented by the linear map

$$\begin{pmatrix} p_1(i+1) \\ p_2(i+1) \end{pmatrix} = \begin{pmatrix} \mp 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} p_1(i) \\ p_2(i) \end{pmatrix}, \qquad (1)$$

where the upper (lower) signs are to be taken in the case of the collision with the walls at $q_{1,\min}$ or $q_{1,\max}$ ($q_{2,\min}$ or $q_{2,\max}$). The two Cartesian momentum coordinates are here represented by p_1 and p_2, and the spatial coordinates are q_1 and q_2. It is clear from Eq. (1) that the entire trajectory of a point particle in this system only visits the four points $(p_1(0),p_2(0))$, $(-p_1(0),p_2(0))$, $(p_1(0),-p_2(0))$, $(-p_1(0),-p_2(0))$ in momentum space, where $p_1(0)$ and $p_2(0)$ are the initial momentum conditions.

We can define the frequencies $\omega_i = |p_i(0)|/(q_{i,\max} - q_{i,\min})$. The trajectory will be periodic in coordinate space, if ω_1/ω_2 is in the set of rational numbers.[6] However, the case of periodic trajectories is irrelevant for practical physical systems with random initial momentum conditions, because the rational numbers are a set of measure 0 within the real numbers. It is of some concern in the computer studies because the computer only deals with rational numbers.

The trajectory of a particle confined to a rectangular box becomes chaotic if a circular scattering center is placed somewhere inside the box. We choose to put our circle at the center of the box and choose its radius R large enough such that the only trajectories which can completely miss the circle are those with $p_2(0)=0$ and $q_2(0)\in(R,q_{2,\max})$ or $p_1(0)=0$ and $q_1(0)\in(R,q_{1,\max})$. This creates a physical situation which is analogous to Sinai's billiard.[1] The resulting trajectories of the particles are chaotic.

The map for reflections off the circle is given by

$$\begin{pmatrix} p_1(i+1) \\ p_2(i+1) \end{pmatrix} = \frac{1}{R^2} \begin{pmatrix} [q_2(i)^2-q_1(i)^2] & [-2q_1(i)q_2(i)] \\ [-2q_1(i)q_2(i)] & [q_1(i)^2-q_2(i)^2] \end{pmatrix} \begin{pmatrix} p_1(i) \\ p_2(i) \end{pmatrix}. \tag{2}$$

This map, as well as the map defined by Eq. (1), is of course area preserving, because in an elastic collision the kinetic energy is conserved.

Since the system described by this map is a K system (positive Kolmogorov entropy, close orbits separate exponentially), the trajectroy is ergodic.[7,8] Therefore the time-averaged phase-space density is constant at every point in the allowed region of phase space, which is a circle in momentum space and the entire coordinate space area of the box excluding scattering center.

In Fig. 1, we present the results of our numerical simulation of the two cases discussed above. In this simulation, we calculated $N(0)=10^6$ events with random initial conditions each for the chaotic and the regular case. The absolute value of the particle's momentum p, the box side length l, and the mass m are irrelevant and only result in a rescaling of the time variable according to $t\propto ml/p$. In both cases we chose $p=2.5$ and $m=1$. The box dimensions in the chaotic case are 12×12 with a circle of radius 4.5 in the center. Then the total coordinate-space area available is $A_c=80.38$. The dimensions of the box without the scattering center are adjusted such that the total area in both cases is identical. We use a quadratic box with side lengths 8.966. Displayed in Fig. 1 are the number of particles which escaped through the hole of width $\Delta=0.2$ in a given time interval, which we binned in units of 60.

For the chaotic case, we find an exponential decay

$$N(t)=N(0)\exp(-t/\tau) \tag{3}$$

with an extracted decay constant $\tau\approx510$. A few simple considerations lead to the analytic calculation of this constant. The number of particles leaving per time interval is given by

$$\dot N(t)=\Delta\rho(t)\int d^2p\,\mathbf{p}\cdot\mathbf{e}_n=-2\Delta\rho(t)p^2\delta p\,, \tag{4}$$

where $\mathbf{e}_n$ is a unit vector normal to the opening in the surface, and the integration in momentum space is taken over a circular ring with radius p and infinitesimal width δp. $\rho(t)$ is the phase-space density of particles. In the case of ergodic motion, it is only a function of time and not of any of the phase-space coordinates. It is in our case

$$\rho(t)=N(t)/\Omega=N(t)/2\pi p\delta pA_c\,. \tag{5}$$

Inserting this into Eq. (4) yields

$$\dot N(t)=(-1/\tau)N(t)\,,\qquad \tau=\pi A_c/p\Delta\,. \tag{6}$$

With the numerical parameters used in the calculation, we obtain a value of $\tau=505.6$, in good agreement with the graphically extracted value.

In contrast to this result, we find for the classical case

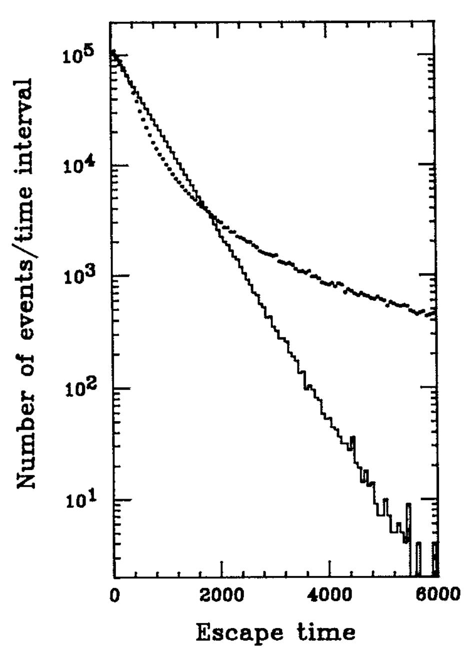

FIG. 1. Differential decay rates for the two classical billiards of the same total area. The exponentially falling histogram represents the chaotic case, and the dots show the results of the regular motion. In both cases, a total number of 10^6 events with random initial conditions as described in the text was calculated.

a power-law decay

$$\lim_{t\to\infty} \dot{N}(t) \propto t^{-\lambda} \quad (7)$$

with a numerically extracted value of $\lambda \approx 2$, independent of the container or window size. This implies that the mean decay time diverges.

In retrospect, the power-law behavior for the regular motion is understandable with the following heuristic argument. Since $|\mathbf{p}\cdot\mathbf{e}_n|$ is a constant of the motion, there will be different characteristic decay times for different parts of phase space, and, in fact, the decay rate is proportional to $|\mathbf{p}\cdot\mathbf{e}_n|$. Within each momentum group the decay should be exponential according to the above arguments, because if we fix the momentum the total phase space is identical to the coordinate space; and the motion is ergodic there. Then the overall decay is given by

$$\begin{aligned} N(t) &\sim \int d^2p \exp(-ct|\mathbf{p}\cdot\mathbf{e}_n|) \\ &\sim \int_{-\pi/2}^{\pi/2} \exp(-ctp\cos\phi)d\phi \\ &\sim t^{-1} \text{ for } t\to\infty . \end{aligned} \quad (8)$$

Here c is some constant independent of $\mathbf{p}$. Then the differential decay law is the derivative, $\dot{N}(t)\sim t^{-2}$, as observed in the model.

Our results can be extended to higher dimensions as well. We chose as the container a D-dimensional hypercube and the scattering center a D-dimensional hypersphere. The window is a $(D-1)$-dimensional hypercube. If we choose the ratio of the window "area" to the container "volume" to be a constant independent of D, then the result for the chaotic scenario remains unchanged, and we obtain an exponential decay with the same decay constant τ.

With the regular hypercube we find a power-law decay with the same power as in the two-dimensional case, according to the considerations above. This last result we verified by numerical calculations in 2,3, . . . ,10 dimensions. Here, we chose the ratio r =window-area/container-surface-area (=probability of hitting the window with a given straight segment of the trajectory) to be a constant independent of D. Then the window side length Δ_d relative to the cube side length L_d (which we also chose to be a constant) is given by

$$\Delta_d = L_d(2rD)^{1/(D-1)} . \quad (9)$$

We performed simulations with 10^6 events with $p=1$ and $L_d=2$ in each case and obtained power-law decays for all numbers of dimensions considered. The numerically extracted decay constants were between -1.95 and -1.85 with typical statistical errors on the order of ± 0.1. In Fig. 2, we display our result for all cases. The histograms in this doubly logarithmic plot are binned in units of 200. In order to be able to visually distinguish the curves, they were displaced by factors of 10 relative to each other. The common power-law decay can be clearly recognized.

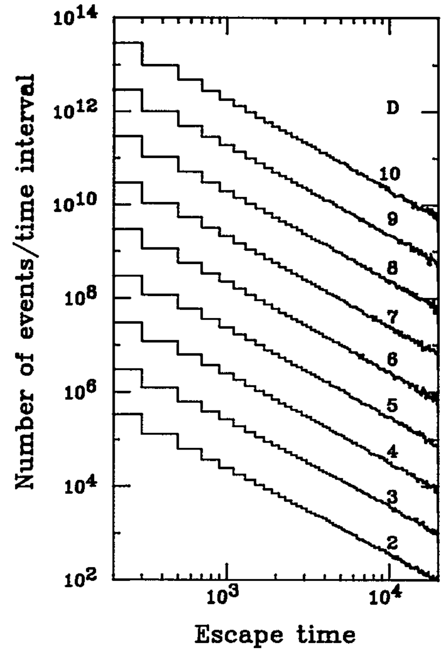

FIG. 2. Differential decay rates for the classical regular billiard in different numbers of dimensions D. The different histograms are multiplied by factors of 10^{D-2} for better visual separation.

In summary, we have considered the simple case of elastic reflections of point particles off container walls in a billiard geometry, and we have obtained different decay laws for the ordered (power-law) and the chaotic (exponential-law) case.

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[1]Ya. G. Sinai, Russ. Math. Surveys **25**, 137 (1970).

[2]O. Haxel, J. H. D. Jensen, and H. E. Suess, Phys. Rev. **75**, 1766 (1949); M. G. Mayer, Phys. Rev. **75**, 1969 (1949).

[3]E. Fermi, Z. Phys. **48**, 73 (1928).

[4]W. Paul, O. Osberghaus, and E. Fischer, Forschungsber. Wirtsch. Verkehrsminist. Nordrhein-Westfalen **415**, 1 (1955); E. Fischer, Z. Phys. **156**, 1 (1959); H. A. Schuessler, E. N. Fortson, and H. G. Dehmelt, Phys. Rev. **187**, 5 (1969).

[5]W. Swiatecki, in *Semiclassical Descriptions of Atomic and Nuclear Collisions,* edited by J. Bang and J. de Boer (North-Holland, Amsterdam, 1985); J. Randrup and W. Swiatecki, Ann. Phys. (N.Y.) **125**, 193 (1980).

[6]In this case, $\omega_1/\omega_2=n_1/n_2$, where n_1 and n_2 are integers with no common factor. Then the trajectory only touches the container walls in $N=2(n_1+n_2)$ points. If we cut a hole of size Δ into one of the container walls, then the total decay probability is given by $\mathcal{P}=1-[(L-\Delta)/L]^N$, where $L=q_{1,\max}-q_{1,\min}+q_{2,\max}-q_{2,\min}$ is the total surface length of

the boundary.

[7]H. G. Schuster, *Deterministic Chaos* (VCH Verlagsgesellschaft, Weinheim, 1988).

[8]B. Misra, I. Prigogine, and M. Courbage [Proc. Natl. Acad. Sci. U.S.A. **76**, 3607 (1979); **76**, 4768 (1979)] have shown that a K system is sufficient for the existence of a Liapunov function, while an ergodic system is not. They have pointed out the close links between intrinsic irreversibility, inherent randomness, and dynamical instabilities. Further references can be found in A. J. Lichtenberg and M. A. Liebermann, *Regular and Stochastic Motion* (Springer-Verlag, Heidelberg, New York, 1982).