

Worksheet #12 - PHY102 (Spr. 2008)
AC Circuits : Transients and resonance
Due Friday April 11th 6pm

In earlier worksheets we have studied the behavior of damped, mass-spring systems. We also took a brief look at the linear and non-linear pendulum problems. The equations describing these problems also describe the behavior of linear and non-linear circuits and there are many analogies between mechanical and electrical problems that are often used in the physics discussions. For example the concepts of underdamped, critically damped and overdamped apply to electrical problems as well as to mechanical problems. Similarly the concept of resonance applies to both mechanical and electrical problems. In circuits, a resistor absorbs energy and acts as the damping. In mechanical problems energy is transferred between potential energy and kinetic energy. In electrical problems energy is transferred between energy stored in the magnetic field and energy stored in the electrical field. In the case of circuits energy is transferred between capacitors and inductors.

The relation between the magnitudes of the voltage and current in circuit elements L,R,C are as follows:

$$V_R = IR, \quad V_L = L \frac{dI}{dt}, \quad V_C = \frac{1}{C} \int_0^t I dt \quad (1)$$

Kirchhoff's voltage law states that the sum of the voltage drops around a series LRC circuit must sum to zero, yielding the following equation,

$$V_s - V_R - V_L - V_C = 0 \quad (2)$$

Using the relations above between the current and voltages for L,R,C given above we find,

$$V_s - IR - L \frac{dI}{dt} - \frac{1}{C} \int_0^t I dt = 0 \quad (3)$$

Taking a derivative of this equation yields,

$$\frac{dV_s}{dt} - L \frac{d^2I}{dt^2} - R \frac{dI}{dt} - \frac{I}{C} = 0 \quad (4)$$

Recall that a damped mass-spring system is described by $F_s - bv - kx = ma$, so that,

$$-F_s + m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0 \quad (5)$$

where F_s is the driving force on the mass-spring system. The analogies between the mass-spring system and the electric circuit are evident from these equations, that is, L acts like the mass, R like the damping and $1/C$ like the spring constant. The derivative of the source voltage acts like the driving force.

Problems

1. (Transients) Consider a series LRC circuit connected to DC voltage of 1V. A switch in the circuit is initially open and the capacitor is initially uncharged. Consider closing the switch at time zero. Using DSolve, find the current in the RLC circuit as a function of time after the switch is closed. The natural frequency when $R = 0$ is $\omega_0 = 1/(LC)^{1/2}$. Using $L = 10mH, C = 1\mu F$ On the same graph, plot the behavior of $i(t)$ for $R = 50, 199, 500$, to illustrate the cases of underdamped, critically damped and overdamped circuits.

2. (Resonance) Consider an LRC circuit driven by a steady sinusoidal AC source with amplitude 1V. Use the same values of L, C as in the first problem. Also set $R = 10$. Use DSolve to find $i(t)$ when an AC voltage $\cos(\omega t)$ is applied to the circuit, instead of the constant voltage of problem 1. In this case, at long times, the current oscillates at the same frequency as the applied voltage. The amplitude and phase of the current depends on the natural frequency of the circuit and the applied frequency. Resonance occurs when they are the same. Make a plot of the current as a function of time for the case $\omega = 50s^{-1}$. Note the time at which the steady state behavior sets in. By using "Table" to make a list of the values of the current in the steady state regime, and then using "Max" to find the maximum current in this regime, plot the dependence of the maximum current in the steady state regime as a function of applied frequency. It is nicer to normalize this to the current at resonance, forming the ratio $i_{max}(\omega)/i_{max}(\omega_R)$. It is easy to find the maximum current at resonance as there, only the resistive part of the impedance remains, so that $i_{max} = V_0/R$ and the phase angle is zero.