PHY481 - Solutions to Homework 4

Problem 4.1 - Solve using a Gaussian surface inside the metal. The electric field everywhere is zero on this surface is zero, therefore the total enclosed charge is zero. This means that the induced charge on the metal surface adjacent the cavity must equal to the charge inside the cavity. This also implies that if there is no charge inside the cavity, the net charge on the metal surface adjacent the cavity is zero.

Problblme 4.2 We solved a two slab problem in lectures using superposition - that method extends to this case. The detailed solution is on closed reserve in the BPS library

Problem 4.9 The real charge q is at position (a, a), while the three image charges are at $(-a, a)$, $(a, -a)$, $(-a, -a)$. The contribution of each of the charges to the potential at position (y, z) is found using cartesian co-ordinates, yielding,

$$V(y,z) = kq[\frac{1}{[(y-a)^2+(z-a)^2]^{1/2}} - \frac{1}{[(y+a)^2+(z-a)^2]^{1/2}}$$

$$+\frac{1}{[(y+a)^2+(z+a)^2]^{1/2}} - \frac{1}{[(y-a)^2+(z+a)^2]^{1/2}}] \tag{1}$$

Each term needs to be expanded for large $r = (y^2+z^2)^{1/2}$. For example the expansion for the first term is

$$\frac{1}{[(y-a)^2+(z-a)^2]} = \frac{1}{r(1-\frac{2a(y+z)}{r^2}+\frac{2a^2}{r^2})^{1/2}} \tag{2}$$

Then use $1/(1+\delta)^{1/2} = 1-\delta/2+3\delta^2/8+....$, with $\delta = -\frac{2a(y+z)}{r^2}+\frac{2a^2}{r^2}$. Note that δ is different for each of the four terms in the potential $V(y, z)$. When this expansion is carried out for the four terms in the potential function and the results are summed, the leading term in the final result is $V(y, z) = 12kqa^2yz/r^5$. After transforming to polar co-ordinates, this is the same as the answer given in PS. The detailed working of the problem is on closed reserve in the BPS library.

Problem 4.11 again requires expansion of the electrostatic potential. In this case the potential is given by,

$$V(r) = kq[\frac{1}{r-r_0} - \frac{1}{r+r_0} + \frac{2}{[(r+r_0/2)^2+3r_0^2/4]^{1/2}} - \frac{2}{[(r-r_0/2)^2+3r_0^2/4]^{1/2}}] \tag{3}$$

In this case expand using, $1/(1+\delta)^{1/2} = 1 - \delta/2 + 3\delta^2/8 - 15\delta^3/48 + ...$, and $1/(1+\delta) = 1 - \delta + \delta^2 - \delta^3 + ...$ In this problem the first three terms, monopole, dipole, quadrupole, are all zero. Full details are on closed reserve in the BPS library.

Problem 4.14 is relatively straightforward and requires calculating the energy in the electric field and comparing it to the capacitor formula $CV^2/2$ in order to extract C for concentric spheres. The electric field is finite only between the two spheres, where its magnitude is kQ/r^2 the energy in the electric field is given by,

$$U = \int u d^3r = \frac{1}{2}\epsilon_0 \int_a^b 4\pi r^2 (\frac{kQ}{r^2})^2 dr = \frac{1}{2}\epsilon_0 k^2 Q^2 4\pi [\frac{1}{a} - \frac{1}{b}] \tag{4}$$

Equating this result to $U = CV^2/2$, where V is the voltage difference between the shells ie $V = kQ[1/a - 1/b]$ enables us to extract $C = 4\pi\epsilon_0 ab/(b-a)$.

Problem 4.15 involves a point charge near a sphere and the formulas required are found as follows (see also Lecture 12). The potential is the sum of the contributions of the real charge and the image charge,

$$V(r,\theta) = \frac{kq}{r_1} + \frac{kq}{r_2} = k[\frac{q}{(r^2 + z_0^2 - 2rz_0 cos\theta)^{1/2}} + \frac{q'}{(r^2 + z_0'^2 - 2rz_0' cos\theta)^{1/2}}] \tag{5}$$

where as usual we used the cosine rule to find expressions for r_1 and r_2. The conjugate relations are $z_0' = R^2/z_0, q' = -qR/z_0$. Subsituting these expressions in the above equation and simplifying yields,

$$V(r,\theta) = kq[\frac{1}{(r^2 + z_0^2 - 2rz_0 cos\theta)^{1/2}} - \frac{R}{(r^2 z_0^2 + R^4 - 2rz_0 R^2 cos\theta)^{1/2}}] \tag{6}$$

The electric field is given by,

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} + E_\phi\hat{\phi} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}. \tag{7}$$

The component $E_\phi = 0$ as there is no ϕ dependence in V. The other two components are,

$$E_r = -kq[-\frac{r - z_0 cos\theta}{(r^2 + z_0^2 - 2rz_0 cos\theta)^{3/2}} + \frac{rz_0^2 R - z_0 R^3 cos\theta}{(r^2 z_0^2 + R^4 - 2rz_0 R^2 cos\theta)^{3/2}}] \tag{8}$$

and

$$E_\theta = -\frac{kq}{r}[\frac{-rz_0 sin\theta}{(r^2 + z_0^2 - 2rz_0 cos\theta)^{3/2}} + \frac{rz_0 R^3 sin\theta}{(r^2 z_0^2 + R^4 - 2rz_0 R^2 cos\theta)^{3/2}}] \tag{9}$$

a) The force on the charge q is $\vec{F}_q = q\vec{E}_{q'}$ where $\vec{E}_{q'}$ is the electric field of the image charge, the second term of equations (8) and (9) above, evaluated at the location of the real charge q i.e. at $(r,\theta) = (z_0, 0)$, where $E_\theta = 0$ and E_r gives,

$$\vec{E}_{q'} = -kq\frac{(z_0^3R - z_0R^3)}{(z_0^4 + R^4 - 2z_0^2R^2)^{3/2}}\hat{r} = -kq\frac{z_0R(z_0^2 - R^2)}{(z_0^2 - R^2)^3}\hat{r} \tag{10}$$

Which reduces to

$$\vec{E}_{q'} = \frac{-kqz_0R}{(z_0^2 - R^2)^2}\hat{r} \quad \text{so that} \quad \vec{F}_q = \frac{-kq^2z_0R}{(z_0^2 - R^2)^2}\hat{r} \tag{11}$$

b) A plot of this function shows a divergence to negative infinity at $z_0 \to R$.

c) The work required to bring the charge q from infinity to position z_0 is given by,

$$Work = -\int \vec{F}\cdot d\vec{r} = \int_\infty^{z_0} \frac{kq^2z_0'R}{(z_0'^2 - R^2)^2}dz_0' = \frac{kq^2R}{2(z_0^2 - R^2)}. \tag{12}$$

d) The electric field is given above.

Problem 4.16 is solved by using three image charges - the usual image charge for a sphere plus two additional image charges found by reflecting q and q' across the $z = 0$ plane as for a plane geometry problem.

a) The total potential has four terms

$$V(r,\theta) = \frac{kq}{r_1} + \frac{kq'}{r_2} - \frac{kq'}{r_3} - \frac{kq}{r_4} \tag{13}$$

Using the cosine rule and the conjugate relations, yields,

$$V(r,\theta) = kq[\frac{1}{(r^2 + z_0^2 - 2rz_0cos\theta)^{1/2}} - \frac{R}{(r^2z_0^2 + R^4 - 2rz_0R^2cos\theta)^{1/2}}$$
$$-\frac{1}{(r^2 + z_0^2 + 2rz_0cos\theta)^{1/2}} + \frac{R}{(r^2z_0^2 + R^4 + 2rz_0R^2cos\theta)^{1/2}}] \tag{14}$$

b) to find the charge induced on the "boss" we note that $-q = q_{boss} + q_{flat}$, where q_{flat} is on the flat part of the surface. On the flat parts $\sigma_{flat}(r) = -\epsilon_0E_\theta(\theta = \pi/2)$, and we use $E_\theta = (-1/r)(\partial V/\partial\theta)$.

$$E_\theta(r,\theta) = \frac{-kq}{r}[\frac{-rz_0sin\theta}{(r^2 + z_0^2 - 2rz_0cos\theta)^{3/2}} + \frac{rz_0R^3sin\theta}{(r^2z_0^2 + R^4 - 2rz_0R^2cos\theta)^{3/2}}$$
$$-\frac{rz_0sin\theta}{(r^2 + z_0^2 + 2rz_0cos\theta)^{3/2}} + \frac{rz_0R^3sin\theta R}{(r^2z_0^2 + R^4 + 2rz_0R^2cos\theta)^{3/2}}] \tag{15}$$

On the flat regions, $\theta = \pi/2$, which reduces this expression to,

$$E_\theta(\pi/2) = 2kq[\frac{z_0}{(r^2+z_0^2)^{3/2}} - \frac{z_0 R^3}{(r^2 z_0^2 + R^4)^{3/2}}]. \tag{16}$$

The total charge on the flat parts is then,

$$q_{flat} = \int_{R_0}^{\infty} 2\pi r dr (2kq)[\frac{z_0}{(r^2+z_0^2)^{3/2}} - \frac{z_0 R^3}{(r^2 z_0^2 + R^4)^{3/2}}] = \frac{q}{z_0}\frac{R^2 - z_0^2}{(R^2+z_0^2)^{1/2}} \tag{17}$$

The fraction on the boss is

$$q_{boss}/(-q) = (1 - q_{flat}/(-q)) = 1 + \frac{1}{z_0}\frac{R^2 - z_0^2}{(R^2+z_0^2)^{1/2}} \tag{18}$$

Problem 4.19 is solved using a single image and is in close analogy to the case of a point charge outside a conducting sphere. We write,

$$V(r,\theta) = \frac{kq}{r_1} + \frac{kq}{r_2} = k[\frac{q}{(r^2+z_0^2-2rz_0cos\theta)^{1/2}} + \frac{q'}{(r^2+z_0'^2-2rz_0'cos\theta)^{1/2}}] \tag{19}$$

To apply this expression to the cavity problem, we use $z_0 = b, R = a, z_0' = a^2/b, q' = -qa/b$, which yields,

$$V(r,\theta) = kq[\frac{1}{(r^2+b^2-2rbcos\theta)^{1/2}} - \frac{a}{(r^2b^2+a^4-2ra^2bcos\theta)^{1/2}}] \tag{20}$$

b) The charge density on the cavity wall is $\sigma(\theta) = -\epsilon_0 E_r(a,\theta) = \epsilon_0(\partial V/\partial r)(a,\theta)$. Note that the direction of the normal to the cavity is in the negative $\hat{r}$ direction leading to the extra negative sign in this relation. The electric field in the radial direction is,

$$E_r(r,\theta) = kq[\frac{r - bcos\theta}{(r^2+b^2-2rbcos\theta)^{3/2}} - \frac{a(rb^2 - a^2bcos\theta)}{r^2b^2+a^4-2ra^2bcos\theta)^{3/2}}] \tag{21}$$

Evaluating at $r = a$ yields,

$$E_r(a,\theta) = kq[\frac{a - bcos\theta}{(a^2+b^2-2abcos\theta)^{3/2}} - \frac{a^2b(b - acos\theta)}{a^3(b^2+a^2-2abcos\theta)^{3/2}}] \tag{22}$$

simplifying this expression and multiplying by ϵ_0 gives,

$$\sigma(\theta) = \frac{q(b^2-a^2)}{4\pi a(a^2+b^2-2abcos\theta)^{3/2}} \tag{23}$$

c) The total charge on the cavity wall is found using Gauss's law (easy) of by integration (hard). Using integration we have,

$$q_{cav} = \int_0^{\pi} 2\pi a^2 sin\theta\sigma(\theta)d\theta = \int_0^{\pi} \frac{aq(b^2-a^2)sin\theta d\theta}{2(a^2+b^2-2abcos\theta)^{3/2}}$$

$$= \frac{-q(a^2-b^2)}{2b(a^2+b^2-2abcos\theta)^{1/2}}|_0^\pi = \frac{-q(a^2-b^2)}{2b}[\pm\frac{1}{a+b} - \pm\frac{1}{a-b}] = \pm q \tag{24}$$

The physical solution is $-q$, as is deduced using Gauss's law following the procedure of problem 4.1.

Problem 4.23 is analogous to the grounded cylinder problem.

a) The electrostatic potential is found using one image line of charge density $-\lambda$,

$$V(r,\phi) = \frac{\lambda}{2\pi\epsilon_0}ln(c_1/r_1) - \frac{\lambda}{2\pi\epsilon_0}ln(c_2/r_2) + c_3 \tag{25}$$

Using the cosine rule we find,

$$V(r,\phi) = \frac{\lambda}{4\pi\epsilon_0}[ln(r^2+x_0'^2-2rx_0'cos\phi) - ln(r^2+x_0^2-2rx_0cos\phi)] + c_4 \tag{26}$$

Using the conjugate relation, $x_0' = R^2/x_0$ and simplifying yields,

$$V(r,\phi) = \frac{\lambda}{4\pi\epsilon_0}[ln(d^2r^2+R^4-2rdR^2cos\phi) - ln(r^2+d^2-2rdcos\phi)] + c_5 \tag{27}$$

b) The electric field in the ϕ direction $E_\phi(R,\phi) = 0$. The charge density on the cylinder is $\sigma(\phi) = \epsilon_0\partial V/\partial r$ where the sign is positive due to the fact that the normal to the cavity surface is in the $-\hat{r}$ direction. From the potential, we find,

$$\frac{\partial V}{\partial r} = \frac{\lambda}{4\pi\epsilon_0}[\frac{2rd^2-2R^2dcos\phi}{d^2r^2+R^4-2rdR^2cos\phi} - \frac{2r-2dcos\phi}{r^2+d^2-2rdcos\phi}]. \tag{28}$$

We then find,

$$\sigma(\phi) = \epsilon_0\frac{\partial V}{\partial r}|_R = \frac{\lambda}{2\pi R}[\frac{R^2-d^2}{R^2+d^2-2dRcos\phi}] \tag{29}$$

c) To find the total charge per unit length on the cavity wall, we can use Gauss's law (easy) or integration (hard). Using integration, we have,

$$\lambda_{cav} = \int_0^{2\pi} Rd\phi\sigma(\phi) = \int_0^{2\pi}\frac{\lambda(R^2-d^2)d\phi}{2\pi(R^2+d^2-2dRcos\phi)} \tag{30}$$

To carry out the integral, use,

$$\int_0^{2\pi}\frac{d\theta}{a+bcos\theta} = \frac{2}{(a^2-b^2)^{1/2}}Tan^{-1}[\frac{(a-b)tan(\theta/2)}{(a^2-b^2)^{1/2}}] \tag{31}$$

where $a = R^2+d^2, b = -2dR$. We then find,

$$\lambda_{cav} = \frac{\lambda(R^2-d^2)}{2\pi}\frac{2}{R^2-d^2}[Tan^{-1}(\frac{(R-d)^2}{R^2-d^2}tan\pi) - Tan^{-1}(\frac{(R-d)^2}{R^2-d^2}tan0)] \tag{32}$$

where we use the fact that $(a^2 - b^2)^{1/2} = ((R^2 + d^2)^2 - 4d^2R^2)^{1/2} = R^2 - d^2$, and $a - b = R^2 + d^2 - 2dR = (R - d)^2$. Both $tan\pi$ and $tan0$ are zero, and $tan^{-1}0$ has solutions $0, \pm\pi, \pm 2\pi$.... i.e. at all angles where $sinx = 0$. We can therefore find solutions $\lambda_{cav} = \pm nq$, where $n = 0, \pm 1, \pm 2$..... The physical solution as deduced using Gauss's law is $\lambda_{cav} = -q$.

d) The force per unit length on the line charge is found by taking the electric field due to the image charge, evaluated at $r = d$, $\phi = 0$ and noting that it acts in the negative $\hat{r}$ direction, so that

$$\vec{F}_\lambda = -\lambda E'_r(d,0)\hat{r} = \frac{-\lambda^2}{4\pi\epsilon_0}[\frac{2d^3 - 2R^2d}{d^4 + R^4 - 2d^2R^2}\hat{r} = -(\pm)\frac{\lambda^2 d}{(2\pi\epsilon_0(d^2 - R^2)}\hat{r}. \tag{33}$$

Notice again that in the surd we can choose either the positive or negative sign. The correct direction is the positive $\hat{x}$ direction, which corresponds to the positive sign in the surd.

Problem 4.24 can be solved directly by adding the electric fields due to the real charge and the image charge, or by using the formula in Lecture 12, to find $E_\theta = 0$, $E_r = (kq/a^2)(1/16 - 2/25)$. Using the first method, we have that the magnitude of E is the sum of the electric field of the real charge and that of the image so that,

$$\vec{E}(0,0,-2a) = [\frac{kq}{(4a)^2} + \frac{kq'}{(2a + z'_0)^2}](-\hat{k}) \tag{34}$$

we also have $z'_0 = R^2/z_0 = a^2/2a = a/2$, and $q' = -qR/z_0 = -q/2$, Substitution into the above equation gives, $\vec{E}(0,0,-2a) = 7kq/(400a^2)\hat{k}$.