Hints for homework 5

Chapters 5,6 of PS

5.7. This is a series expansion problem similar to Example 1 of PS. The difference is that now both the cosh and sinh terms are kept as the problem is neither purely odd or even in the y-direction. The relation between the coefficients of the cosh and sinh terms is found from the condition $V(x,-a/2) = 0$ and the condition $V(x,a/2) = V_0$ is used for the Fourier analysis. An alternative method is to use superposition where we write the problem as a sum of two problems: one with $V_0/2, -V_0/2$ boundary conditions in the y-direction and the other with $V_0/2, V_0/2$ boundary conditions in the y-direction. Both have zero boundary conditions in the x-direction. The latter is a little nicer from a physical perspective. Using the first method, we have,

$$V(x,y) = \sum_{n=0}^{\infty} cos((2n+1)\frac{\pi x}{a})[A_n cosh((2n+1)\frac{\pi y}{a}) + B_n sinh((2n+1)\frac{\pi y}{a})] \tag{1}$$

Then,

$$V(x,-a/2) = A_n cosh((2n+1)\frac{\pi}{2}) - B_n sinh((2n+1)\frac{\pi}{2}) = 0 \quad \text{or} \quad \frac{A_n}{B_n} = tanh[(2n+1)\frac{\pi}{2}] \tag{2}$$

Plugging this into Eq. (1) gives,

$$V(x,y) = \sum_{n=0}^{\infty} cos((2n+1)\frac{\pi x}{a}) B_n [tanh((2n+1)\frac{\pi}{2}) cosh((2n+1)\frac{\pi y}{a}) + sinh((2n+1)\frac{\pi y}{a})] \tag{3}$$

Evaluating this expression at the boundary $y = a/2$ gives,

$$V(x,a/2) = V_0 = \sum_{n=0}^{\infty} cos((2n+1)\frac{\pi x}{a}) B_n [2 sinh((2n+1)\frac{\pi}{2})] = \sum_{n=0}^{\infty} B'(n) cos((2n+1)\frac{\pi x}{a} \tag{4}$$

where $B'(n) = 2B(n) sinh((2n+1)\frac{\pi}{2})$. The Fourier analysis of this expression is exactly the same as that of Example 1 of PS so we find (see Eq. 5.29 of PS).

$$B(n) = \frac{4V_0(-1)^n}{\pi(2n+1)} \frac{1}{2 sinh((2n+1)\frac{\pi}{2})} \tag{5}$$

Substituting this expression into Eq. (3) gives,

$$V(x,y) = \sum_{n=0}^{\infty} \frac{2V_0(-1)^n}{\pi(2n+1)} cos((2n+1)\frac{\pi x}{a})[\frac{cosh((2n+1)\frac{\pi y}{a})}{cosh((2n+1)\frac{\pi}{2})} + \frac{sinh((2n+1)\frac{\pi y}{a})}{sinh((2n+1)\frac{\pi}{2})}] \tag{6}$$

Taking a common denominator and using the identities $sinh(2A) = 2 sinhA coshA$, and $sinh(A+B) = sinh(A)cosh(B) + cosh(A)sinh(B)$ yields the result quoted in PS i.e.

$$V(x,y) = \sum_{n=0}^{\infty} \frac{4V_0(-1)^n}{\pi(2n+1)} cos((2n+1)\frac{\pi x}{a}) \frac{sinh[(2n+1)(\frac{\pi y}{a} + \frac{\pi}{2})]}{sinh((2n+1)\pi)} \tag{7}$$

5.8. a) This problem is solved by making a superposition of the solutions to problem 5.7, yielding,

$$V(x,y) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} [V_2 cos((2n+1)\frac{\pi x}{a}) \frac{sinh[(2n+1)(\frac{\pi y}{a}+\frac{\pi}{2})]}{sinh((2n+1)\pi)}$$

$$+V_1 cos((2n+1)\frac{\pi y}{a}) \frac{sinh[(2n+1)(\frac{\pi x}{a}+\frac{\pi}{2})]}{sinh((2n+1)\pi)} + V_3 cos((2n+1)\frac{\pi y}{a}) \frac{sinh[(2n+1)(-\frac{\pi x}{a}+\frac{\pi}{2})]}{sinh((2n+1)\pi)}] \quad (8)$$

b) To find the value at the origin, evaluate the result of a) at (0,0), yielding,

$$V(0,0) = \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} [(V_1+V_2+V_3) \frac{sinh[(2n+1)(\frac{\pi}{2})]}{sinh((2n+1)\pi)}] = \frac{1}{4}(V_1+V_2+V_3) \quad (9)$$

where we used,

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{(2n+1)} \frac{sinh((2n+1)\pi/2)}{sinh((2n+1)\pi)} = \frac{\pi}{4}. \quad (10)$$

5.9. Only one term is needed here, the $l = 0$ term in the spherical polar expansion $\sum(A_l r^l + B_l/r^{l+1})P_l(cos\theta)$, so $V(r,\theta) = A_0 + B_0/r$. The boundary conditions then give the two equations,

$$V_1 = A_0 + B_0/R_1; \quad V_2 = A_0 + B_0/R_2 \quad (11)$$

Solving these yields,

$$A_0 = V_1 - \frac{R_2(V_1-V_2)}{R_2-R_1}; \quad B_0 = \frac{R_1 R_2 (V_1-V_2)}{R_2-R_1} \quad (12)$$

and the potential,

$$V(r,\theta) = V_1 + \frac{R_2(V_1-V_2)}{R_2-R_1}(\frac{R_1}{r}-1) \quad (13)$$

5.10. a) The first non-trivial case is $P_2(u)$, as no recursion is required for $P_0(u) = 1$, and $P_1(u) = u$. Using the recurrence relation for $P_2(u)$, we find that $C_2 = -3C_0$, and $P_2(u) = C_0 - 3C_0 u^2$. The normalization condition $P_2(1) = 1$ implies that $C_0 = -1/2$, so we find $P_2(u) = (3u^2 - 1)/2$. b) Using Mathematica, we get,

$$P_{10}(u) = \frac{1}{256}(-63 + 3465u^2 - 30030u^4 + 90090u^6 - 109395u^8 + 46189u^{10}); \quad (14)$$

and

$$P_{11}(u) = \frac{1}{256}(-693u + 15015u^3 - 90090u^5 + 218790u^7 - 230945u^9 + 88179u^{11}) \quad (15)$$

c) To prove orthogonality, consider Legendre's equation for two polynomials P_m and P_n as follows,

$$P_n[(1-u^2)P_m'' - 2uP_m' + m(m+1)P_m] = 0; \quad P_m[(1-u^2)P_n'' - 2uP_n' + n(n+1)P_n] = 0 \quad (16)$$

where the primes indicate a derivative with respect to u. Subtracting the first from the second of these two equations and using $d/du(P_nP_m' - P_mP_n') = P_nP_m'' - P_mP_n''$ gives,

$$(1-u)^2\frac{d}{du}(P_nP_m' - P_mP_n') - 2u(P_nP_m' - P_mP_n') + (m(m+1) - n(n+1))P_nP_m = 0 \quad (17)$$

which is equal to,

$$\frac{d}{du}[(1-u^2)(P_nP_m' - P_mP_n')] + (m(m+1) - n(n+1))P_nP_m = 0 \quad (18)$$

Now we integrate this equation over the interval $[-1,1]$. This integral produces zero for the first term of the equation, so the second must also be zero. Therefore if $m \neq n$ the integral of P_nP_m over this interval must be zero, proving othogonality. d) Integrating Legendre's equation, we have,

$$(1-u^2)\frac{dP_l}{du}|_0^1 = -l(l+1)\int_0^1 P_l(u)du \quad \text{or} \quad \int_0^1 P_l(u)du = \frac{P_l'(0)}{l(l+1)} \quad (19)$$

5.15. The general solution in cylindrical co-ordinates, when there is no z-dependence, is given by,

$$V(r,\phi) = (a+b\phi)(c+dln(r)) + \sum_{n=1}^{\infty}(A_nr^2 + \frac{B_n}{r^n})(C_ncos(n\phi) + D_nsin(n\phi)) \quad (20)$$

The boundary conditions are odd in ϕ, so we choose $C_n = 0$. We have to construct different solutions on the interior and the exterior. The exterior solution must converge as $r \to \infty$ so we set $A_n \to 0$ and $c = d = 0$, so that,

$$V_{ext}(r,\phi) = \sum_{n=1}^{\infty}\frac{b_n}{r^n}sin(n\phi)) \quad (21)$$

we also need to ensure that the potential is positive for $\phi\varepsilon[0,\pi]$ and negative for $\phi\varepsilon[0,-\pi]$, which is achieved by choosing only odd values of n in the sum above, so that,

$$V_{ext}(r,\phi) = \sum_{n=odd}^{\infty}\frac{b_n}{r^n}sin(n\phi)) \quad (22)$$

Now we impose the boundary condition on the voltage, by evaluating at $r = R$, multiplying both sides by $sin(m\phi)$ and integrating over $[0,2\pi]$ so that,

$$\int_0^{\pi} V_0sin(m\phi)d\phi + \int_{\pi}^{2\pi}(-V_0)sin(m\phi)d\phi = \int_0^{2\pi}\sum_{n=odd}^{\infty}\frac{b_n}{R^n}sin(n\phi))sin(m\phi)d\phi = \frac{\pi b_m}{R^m} \quad (23)$$

for m odd. Notice that for m even the LHS is zero confirming our intuitive choice of m odd. Carrying out the integrals gives,

$$\frac{-2V_0}{m}[cos(m\pi) - cos(0)] = \frac{4V_0}{m} = \frac{\pi b_m}{R^m}; \quad \text{or} \quad b_n = \frac{4V_0 R^n}{\pi n} \tag{24}$$

Therefore,

$$V_{ext}(r,\phi) = \sum_{n=odd}^{\infty} \frac{4V_0}{\pi n}\frac{R^n}{r^n} sin(n\phi) \tag{25}$$

The calculation for the interior solution is the same except that we take $B_n = 0$ to find,

$$V_{int}(r,\phi) = \sum_{n=odd}^{\infty} \frac{4V_0}{\pi n}\frac{r^n}{R^n} sin(n\phi) \tag{26}$$

5.17. This is a bit different as we are given the charge density at the surface. The charge density is related to the electric field, through

$$E_r(R_+,\phi) - E_r(R_-,\phi) = \frac{\sigma(\phi)}{\epsilon_0}, \quad E_\phi(R_+,\phi) = E_\phi(R_-,\phi) \tag{27}$$

On approach to the surface you can think of this as similar to a flat sheet of uniform charge density where these relations hold. Alternatively we can use the differential form of Maxwell's equations $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ to demonstrate these relations. From these relations, we see that the potential needs to have angular symmetry that is the same as that of the charge density. Since the symmetry of the charge density is even when $\phi \to -\phi$, we need to take the cosine series for the potential, so the external and internal potentials are given by,

$$V_{int}(r,\phi) = \sum_n a_n r^n cos(n\phi); \quad V_{ext}(r,\phi) = \sum_n \frac{b_n}{r^n} cos(n\phi) \tag{28}$$

The electric field is given by, $E_r = -\partial V/\partial r$, $E_\phi = (-1/r)(\partial V/\partial\phi)$, so the boundary condition $E_\phi(R_+,\phi) = E_\phi(R_-,\phi)$ implies that,

$$\sum_n a_n R^n sin(n\phi) = \sum_n \frac{b_n}{R^n} sin(n\phi) \quad \text{or} \quad b_n = R^{2n} a_n \tag{29}$$

The boundary condition on the radial electric field is,

$$E_r(R_+,\phi) - E_r(R_-,\phi) = \frac{\sigma}{\epsilon_0}, \quad \text{so} \quad \sum_n a_n n[R^{n-1} + \frac{R^{2n}}{R^{n+1}}]cos(n\phi) = \frac{\sigma(\phi)}{\epsilon_0} \tag{30}$$

The Fourier analysis is then,

$$\int_0^{\pi/2} \frac{\sigma_0}{\epsilon_0} cos(m\phi)d\phi - \int_{\pi/2}^{3\pi/2} \frac{\sigma_0}{\epsilon_0} cos(m\phi)d\phi + \int_{3\pi/2}^{2\pi} \frac{\sigma_0}{\epsilon_0} cos(m\phi)d\phi = 2\pi m a_m R^{m-1} \tag{31}$$

Evaluating the LHS gives,

$$\frac{\sigma_0}{\epsilon_0 m}[sin(\frac{m\pi}{2}) - sin(\frac{3\pi m}{2}) + sin(\frac{m\pi}{2}) - sin(\frac{3\pi m}{2})] = \frac{2\sigma_0}{\epsilon_0 m}[sin(\frac{m\pi}{2}) - sin(\frac{3\pi m}{2})] \quad (32)$$

The RHS is zero when m is even, which we could have deduced from the outset. When m is odd we write $m = (2j + 1)$ so that $sin(\frac{(2j+1)\pi}{2}) - sin(\frac{3\pi(2j+1)}{2}) = 2(-1)^j$. We then find,

$$a_j = \frac{4\sigma_0(-1)^j}{2\pi\epsilon_0(2j+1)^2 R^{j-1}}, \quad \text{so that} \quad V_{int} = \sum_j \frac{2\sigma_0 R(-1)^j}{\pi\epsilon_0(2j+1)^2}(\frac{r}{R})^{2j+1} cos((2j+1)\phi) \quad (33)$$

The exterior solution is found in the same way, yielding the same solution other than $r/R \to R/r$ in the above equation.

5.32. This is a spherical co-ordinate problem where only the $l = 1$ term is needed. It is easy to show that,

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}(-\frac{pr}{R^3} + \frac{p}{r^2})cos\theta \quad (34)$$

satisfies the boundary conditions, namely a dipole at the origin and zero potential at the conducting sphere, i.e. $V(R,\theta) = 0$.

5.34. a) This problem can be simplified using superposition to write the boundary condition as a sum of $V_0/2$ over the whole circle plus $-V_0/2$ on the upper part and $V_0/2$ on the lower part. The latter problem is odd in ϕ, so the exterior solution is of the form,

$$V_{ext}(r,\phi) = \frac{V_0}{2} + \sum_{n=1}^{\infty} \frac{b_n}{r^n} sin(n\phi) \quad (35)$$

The Fourier analysis then gives,

$$-\int_0^{\pi} \frac{V_0}{2} sin(m\phi)d\phi + \int_{\pi}^{2\pi} \frac{V_0}{2} sin(m\phi) = \frac{\pi b_m}{R^m} \quad (36)$$

The LHS gives $V_0[cos(m\pi) - cos(0)]/m$ which is zero for m even and $-2V_0/m$ for m odd, so that,

$$V_{ext}(r,\phi) = \frac{V_0}{2} - \sum_{n\ odd}^{\infty} \frac{2V_0}{\pi n}(\frac{R}{r})^n sin(n\phi) \quad (37)$$

The interior solution is found by making the substitution $R/r \to r/R$ in this expression. b) At the origin, $V = V_0/2$. c) The electric field at is given by,

$$E_r = \sum_{n\ odd}^{\infty} \frac{2V_0}{\pi n}\frac{n}{r}(\frac{r}{R})^n sin(n\phi); \quad E_\phi = \frac{1}{r}\sum_{n\ odd}^{\infty} \frac{2V_0}{\pi n} n(\frac{r}{R})^n cos(n\phi) \quad (38)$$

At the origin, we then find

$$E_r(0,\phi) = \frac{2V_0}{\pi}\frac{1}{r}(\frac{r}{R})sin(\phi) = \frac{2V_0}{\pi R}sin(\phi) \tag{39}$$

and

$$E_\phi(0,\phi) = \frac{2V_0}{\pi}\frac{1}{r}(\frac{r}{R})cos(\phi) = \frac{2V_0}{\pi R}cos(\phi) \tag{40}$$

Using $\hat{j} = sin\phi\hat{r} + cos\phi\hat{\phi}$, the electric field reduces to $\vec{E} = 2V_0\hat{j}/(\pi R)$

6.3. a) Plot

$$< p_z >= p[-\frac{1}{a} + coth(a)] \tag{41}$$

where

$$a = \frac{pE_0}{k_BT} = \frac{(3.34\times 10^{-30}Cm)(10^6V/m)}{(1.38 * 10^{23}J/K)T} = \frac{0.242K}{T} \tag{42}$$

where T is in Kelvin as a function of T. Also plot the polarizability

$$\alpha = \frac{d < p_z >}{dE_0} = (\frac{1}{a^2} - \frac{1}{[sinh(a)]^2})\frac{p}{k_BT} \tag{43}$$

as a function of T. A good range to plot over is $0.01K < T < 2K$

6.5 a) The bound charge at the surface found from the polarization at the surface, using $\sigma_b = \hat{n}\cdot\vec{P}$. At the top surface we have $\sigma_b = P$, while at the bottom surface $\sigma_b = -P$. The electric field lines look like those coming from a set of dipoles. b) The electric field inside the slab is like that between two equal and opposite sheets of charge at the slab surfaces, with sheet charge density P at the top and $-P$ at the bottom. The electric field in the center is $\vec{E} = -(P/\epsilon_0)\hat{k}$. c) The electric field on the axis of the slab, at large distances from the slab, is found using the formula for a dipole field with the dipole moment equal to the polarization times the volume of the slab, $V = \pi(10h)^2h = 100\pi h^3$. We are asked to find the field at $\vec{r} = 100h\hat{i}$, so that,

$$\epsilon_0\vec{E} = \frac{3V(\vec{P}\cdot\hat{r})\hat{r} - V\vec{P}}{4\pi r^3} = \frac{-VP\hat{k}}{4\pi(100h)^3} = -0.25\times 10^{-4}P\hat{k} \tag{44}$$