H6 - Hints

6.7 This is a capacitor type problem involving two concentric spheres with a dielectric between them. The charge on the capacitor surfaces is $Q, -Q$. Take Q on the inner surface and find the displacement field using Gauss's law, a)

$$\oint \vec{D} \cdot d\vec{A} = Q \quad \text{so that} \quad \vec{D}(r) = \frac{Q}{4\pi r^2}\hat{r} \tag{1}$$

b) The relation between electric field and displacement field $\vec{D} = \epsilon\vec{E}$ and we have

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = (1 - \frac{\epsilon_0}{\epsilon})\vec{D} = (1 - \frac{1.5a - 0.5r}{a})\frac{Q}{4\pi r^2}\hat{r} = \frac{Q(r-a)}{8\pi r^2 a}\hat{r} = P_r\hat{r} \tag{2}$$

The bound charge density is then (using the polar form of the divergence),

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial(r^2 P_r)}{\partial r} = -\frac{Q}{8\pi a r^2} \tag{3}$$

c) The energy is $(1/2)\int \vec{D} \cdot \vec{E} d^3r$, and using $\vec{D} = \epsilon\vec{E}$ yields,

$$U = \int_a^{2a} \frac{D^2}{2\epsilon} 4\pi r^2 dr = \frac{Q^2}{8\pi\epsilon_0 a}\int_a^{2a} \frac{3a - r}{2r^2} d^3r = \frac{Q^2}{8\pi\epsilon_0 a}(\frac{3}{4} - \frac{1}{2}ln(2)) \tag{4}$$

d) Using $U = Q^2/2C$, yields $C = a/(k(0.75 - 0.5ln(2))$.

6.8 The Clausius-Mossotti formula is,

$$\alpha = \frac{3\epsilon_0}{n}\frac{\kappa - 1}{\kappa + 2} \tag{5}$$

a) Solving for κ gives,

$$\kappa = \frac{1 + 2\alpha n/(3\epsilon_0)}{1 - \alpha n/(3\epsilon_0)} \tag{6}$$

There is a divergence in the dielectric constant when $n = 3\epsilon_0/\alpha$. b) The molecular density of water is

$$n = \frac{\rho}{m} = \frac{10^3 kg/m^3}{10 g/mole}\frac{6.02 \times 10^{23} molecules}{mole} = 3.34 \times 10^{28}/m^3 \tag{7}$$

The relation between α and the dipole moment is $\alpha = p_0^2/k_BT$, so $p_0 = (\alpha k_B T)^{1/2}$. Plugging the numbers gives, $p_0 = 3.08 \times 10^{-30} Cm = 0.36 e a_B$.

6.14 This is a parallel plate capacitor problem where the dielectric medium consists of two different dielectrics slabs with dielectric constants κ_1, κ_2 connected in series. a) Using

Gauss's law, the displacement field in both dielectrics is found to be $\vec{D} = -\sigma\hat{k}$. b) $\vec{E} = \vec{D}/\epsilon$, therefore in the region where $\epsilon = \kappa_1\epsilon_0$, $\vec{E} = \sigma/\epsilon_0\kappa_1$, while in the other region, $\vec{E} = \sigma/\epsilon_0\kappa_2$. c) We have,

$$\vec{P} = \vec{D} - \epsilon_0\vec{E} = (1 - \frac{\epsilon_0}{\epsilon})\vec{D} \tag{8}$$

At the top surface, $\sigma_1 = \hat{k}\vec{P}_1 = -(1-1/\kappa_1)\sigma$, at the bottom surface $\sigma_2 = -\hat{k}\vec{P}_2 = (1-1/\kappa_2)\sigma$ At the interior surface we have to add the bound charges of the two components, $\sigma_m = \hat{k}\vec{P}_2 - \hat{k}\vec{P}_1 = (1/\kappa_2 - 1/\kappa_1)\sigma$.

d) The capacitance is given by the series capacitance rule,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d/2}{\epsilon_0\kappa_1 A} + \frac{d/2}{\epsilon_0\kappa_2 A} \tag{9}$$

so that,

$$C = \frac{2A\epsilon_0}{d}\frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2} \tag{10}$$

6.19 a) This is an image charge problem. The force on the real charge is given by $\vec{F} = q\vec{E}_{q'}(d)$, where d is the location of a charge q on the z-axis. We assume that the lower half plane is filled with a dielectric, with dielectric constant ϵ. The electrostatic potential due to the image charge is given by,

$$V_{q'} = -\frac{\kappa - 1}{\kappa + 1}\frac{kq}{r} = V_{q'} = -\frac{\kappa - 1}{\kappa + 1}\frac{kq}{(x^2 + y^2 + (z + d)^2)^{1/2}} \tag{11}$$

where we have taken the dielectric to lie in the low half plane. The electric field at $(0, 0, d)$ is in the $\hat{k}$ direction, so the force on the charge q is given by,

$$\vec{F} = q\frac{-\partial V}{\partial z}|_d = -\frac{\kappa - 1}{\kappa + 1}\frac{kq^2}{4d^2} \tag{12}$$

b) The surface charge density is found from,

$$E_z^{outside}(x, y, 0) - E_z^{inside}(x, y, 0) = \frac{\sigma_b(x, y)}{\epsilon_0} \tag{13}$$

where,

$$E_z^{outside} = \frac{kq(z - d)}{(x^2 + y^2 + (z - d)^2)^{3/2}} - \frac{\kappa - 1}{\kappa + 1}\frac{kq(z + d)}{(x^2 + y^2 + (z + d)^2)^{3/2}} \tag{14}$$

and

$$E_z^{inside} = \frac{2}{\kappa + 1}\frac{kq(z - d)}{(x^2 + y^2 + (z - d)^2)^{3/2}} \tag{15}$$

Evaluating these expressions at $z = 0$, and using $r^2 = x^2 + y^2 + d^2$ gives,

$$\sigma(x, y) = -\frac{\kappa - 1}{\kappa + 1}\frac{2dq}{4\pi r^3} \tag{16}$$

6.24 This is a cylindrical capacitor problem where we need to find the largest possible voltage, V_{max} across the capacitor. Define the charge per unit length on the capacitor to be λ, so the electric field inside the dielectric (From Gauss's law and $\vec{D} = \epsilon\vec{E}$) is,

$$\vec{E} = \frac{\lambda}{2\pi\epsilon r}\hat{r} \tag{17}$$

The largest electric field occurs at the smallest r, so we find λ_{max} the largest charge density the capacitor can hold through,

$$\frac{\lambda_{max}}{2\pi\epsilon r} = 40MV/m; \quad \text{so that} \quad \lambda_{max} = (2\pi)(40MV/m)(\epsilon)(0.3cm) \tag{18}$$

The maximum voltage across the capacitor is found from,

$$V_{max} = \int_{0.3cm}^{0.8cm} E(r)dr = \int_{0.3cm}^{0.8cm} \frac{\lambda_{max}}{2\pi\epsilon r}dr = (40MV/m)(0.3cm)ln(8/3) = .118MV \tag{19}$$

6.31 This is a boundary value problem that can be solved in a similar manner to the case of a dielectric sphere in a constant electric field. In this case the electric field is in the $\hat{i}$ direction, the cylinder radius is a and the cylinder has dielectric constant ϵ. Since the dielectric sphere is uniform, Laplace's equation holds inside and outside, so we have,

$$V_{int} = -Arcos\phi = -Az; \quad V_{ext} = -E_0 rcos\phi + \frac{Ba^2}{r}cos\phi \tag{20}$$

Continuity of the potential and the normal displacement field at the surface yield,

$$B - E_0 = -A; \quad \epsilon A = \epsilon_0(E_0 + B); \quad \text{so that} \quad A = \frac{2E_0}{\kappa+1}; \quad B = \frac{E_0(\kappa-1)}{\kappa+1} \tag{21}$$

a) the electric field inside the cylinder is then

$$\vec{E}_{int} = -\frac{\partial V_{int}}{\partial z}\hat{k} = \frac{2E_0}{\kappa+1}\hat{k} \tag{22}$$

b) The dipole moment per unit length, p_{dul} is found by noting that the "dipole" term due to two parallel line charges λ, $-\lambda$ separated by distance d, is given by, $p_{dul}cos\phi/(2\pi\epsilon_0 r^2)$, where $p_{dul} = d\lambda$. We can then write,

$$\frac{p_{dul}cos\phi}{2\pi\epsilon_0 r^2} = \frac{(\kappa-1)E_0 a^2 cos\phi}{(\kappa+1)r^2} \quad \text{so that} \quad p_{dul} = \frac{2\pi\epsilon_0(\kappa-1)E_0 a^2}{(\kappa+1)} \tag{23}$$