PHY481 - Outline of solutions to Homework 7

7.3 The charge density is $\rho = Q/V = 3Q/(4\pi a^3)$. The current density is given by,

$$\vec{j} = nq\vec{v} = \rho\vec{v} = \rho\vec{\omega}\wedge\vec{r} = \rho\omega(-y\hat{i} + x\hat{j}) \tag{1}$$

In polars,

$$x = r sin\theta cos\phi; \quad y = r sin\theta sin\phi; \quad \hat{\phi} = -sin\phi\hat{i} + cos\phi\hat{i} \tag{2}$$

so that,

$$\vec{j} = \frac{3\omega Q}{4\pi a^3} r sin\theta\hat{\phi} \tag{3}$$

7.16 Power $= VI$, $I = j \times Area = 1802A$. Rate $= I/Q \approx 90/s$.

8.4 a), b) draw pictures. c) $\vec{F} = q\vec{v}\wedge\vec{B} = qv_0 b(-x\hat{i} + y\hat{j}) = m\vec{a}$. which gives the equations,

$$m\frac{d^2x}{dt^2} = -qv_0bx; \quad m\frac{d^2y}{dt^2} = qv_0by; \quad m\frac{d^2z}{dt^2} = 0 \tag{4}$$

Solving the three equations gives, $x(t) = Asin(\omega t) + Bcos(\omega t)$, $y(t) = Ce^{-\omega t} + De^{\omega t}$, $z(t) = z_0 + v_0 t$, where $\omega^2 = qv_0b/m$. The initial conditions are $x(0) = x_0, v_x(0) = 0, y(0) = v_y(0) = 0, z(0) = 0$, yielding, $x(t) = x_0 cos(\omega t)$, $t = z/v_0$ and $x(z) = x_0 cos(\omega z/v_0)$.

8.7 The velocity selector equation comes from $q\vec{E} = q\vec{v}\wedge\vec{B}$, which leads to $v = E/B$. The cyclotron orbit formula is found from $mv^2/R = qvB$, so that $S = 2R = 2mv/(qB)$. Combining these equations, we find, $m = qB^2S/(2E)$.

8.12 The field on the z-axis is directed along the z-axis (by symmetry), and the contribution from each edge of the loop is the same. We thus calculate for only one edge, take the z-component and multiply by four. We use the a) Biot-Savart law with,

$$d\vec{l} = dx\hat{i}; \quad \vec{R} = -x\hat{i} + a\hat{j} + z\hat{k}, \text{ so } d\vec{l}\wedge\vec{R} = -z\hat{j} + a\hat{k} \tag{5}$$

Also $R^2 = x^2 + a^2 + z^2$, so the magnetic field is (for one edge),

$$B_z(z) = \int_{-a}^{a} \frac{\mu_0 i}{4\pi}\frac{adx}{(x^2 + a^2 + z^2)^{3/2}} = \frac{2\mu_0 ia^2}{4\pi(a^2 + z^2)(2a^2 + z^2)^{1/2}} \tag{6}$$

b) As $z \to \infty$, this reduces to $2\mu_0 ia^2/(\pi z^3)$. The dipole formula for the magnetic field on the z-axis gives, $\mu_0 m/(2\pi z^3)$. Comparing the two equations we get $m = 4a^2 i$, which is equal to $i \times Area$ as expected.

c) The magnetic field at the center is given by $2^{1/2}\mu_0 i/(\pi a)$, while the field at the center of a circular loop of radius a is $\mu_0 i/(2a)$. The ratio of the fields at the center of the square and circle is then $2^{3/2}/\pi \approx 0.9$.

8.18 The current densities in the inner and outer parts of the coaxial are, $j_{in} = I_0/(\pi a^2)$; $j_{out} = I_0/(\pi(c^2 - b^2))$. Amperes law for a circular loop of radius r gives, $B = \mu i_{encl}/(2\pi r)$. The remaining task is to find i_{encl}. We have to do this in four regimes,

$$i_{encl} = j_{in}\pi r^2, \quad r < a; \qquad I_0, \quad a < r < b; \qquad [I_0 - j_{out}\pi(r^2 - b^2)], \quad b < r < c; \qquad 0, \quad r > c. \tag{7}$$

The field is found by substituting these into , $B = \mu_0 i_{encl}/(2\pi r)$, and the direction is given by the right hand rule.

8.21 a) Using the identity $\vec{\nabla}(f\vec{F}) = \vec{F}\cdot\vec{\nabla}f + f\vec{\nabla}\cdot\vec{F}$, we have,

$$\vec{\nabla}_r \cdot \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\vec{\nabla}_r \cdot [\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}] \tag{8}$$

$$= \frac{\mu_0}{4\pi}\int d^3r'[\frac{\vec{\nabla}_r\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{J}(\vec{r}')\cdot\vec{\nabla}_r(\frac{1}{|\vec{r}-\vec{r}'|})] \tag{9}$$

The first term in this expression is zero as $\vec{J}$ only depends on $\vec{r}'$. Now we change the derivative of the second term on the RHS to one with respect to $\vec{r}'$, so that

$$\vec{\nabla}_r \cdot \vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi}\int d^3r'\vec{J}(\vec{r}')\cdot\vec{\nabla}_{r'}(\frac{1}{|\vec{r}-\vec{r}'|}) \tag{10}$$

Now we use the identity above again (in reverse this time) to find

$$\vec{\nabla}_r \cdot \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'[\frac{\vec{\nabla}_{r'}\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \vec{\nabla}_{r'}\cdot(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|})] \tag{11}$$

The first term is zero due to the continuity equation and the second one is zero as seen by transforming to a surface integral.

b) The Laplacian of $\vec{A}(\vec{r})$ gives,

$$\vec{\nabla}^2\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'[\vec{J}(\vec{r}')\vec{\nabla}_r^2(\frac{1}{|\vec{r}-\vec{r}'|})] \tag{12}$$

Using Eq. (3.4) of PS, this reduces to,

$$\frac{\mu_0}{4\pi}\int d^3r'[\vec{J}(\vec{r}')(-4\pi\delta(\vec{r}-\vec{r}')) = -\mu_0\vec{J}(\vec{r}) \tag{13}$$

8.24 The magnetic dipole formula is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{(3\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} \tag{14}$$

The radial component of the field is $B_r = \vec{B}\cdot\hat{r} = 2(\vec{m}\cdot\hat{r})/r^3$. Writing $\hat{r} = sin\theta\cos\phi\hat{i} + sin\theta\sin\phi\hat{j} + cos\theta\hat{k}$ allows us to find the field. Similarly for the other two components, where we use, $\hat{\theta} = cos\theta\cos\phi\hat{i} + cos\theta\sin\phi\hat{j} - sin\theta\hat{k}$, and $\hat{\phi} = -sin\phi\hat{i} + cos\phi\hat{j}$. To evaluate the field use $latitude = 90 - \theta$, $longitude = \phi$, and the pole angles $\theta_0 = 169, \phi_0 = 109$,

8.28 a) The current density is $j_0 = I_0/(\pi R_0^2)$, so the current enclosed within an Amperian circle of radius r centered on the axis of the wire is, $i(r) = j_0\pi r^2$, for $r < R_0$. For $r > R_0$ the enclosed current is I_0. The magnitude of the magnetic field for $r < R_0$ is then $\mu_0 I_0 r/(2\pi R_0^2)$. The direction is given by the RHR.

b) The current density is $I_0/(\pi(R_0^2 - b^2))$. The magnetic field is a superposition of a wire with no hole minus the magnetic field due to a wire of radius b, centered at $x = a$, so that,

$$\vec{B}_{total} = \frac{\mu_0 I_0}{2\pi(R_0^2 - b^2)}(r\hat{\phi} - r_s\hat{\phi}_s) \tag{15}$$

where r is the distance from the center of the wire, r_s is the distance from the center of the hole, and $\hat{\phi}$, $\hat{\phi}_s$ are the unit vectors in the $\hat{\phi}$ direction for a co-ordinate system centered at the center of the wire and hole respectively. Now note that,

$$r\hat{\phi} = r(-sin\phi\hat{i} + cos\phi\hat{j}) = (-y\hat{i} + x\hat{j}); \quad r_s\hat{\phi}_s = r(-sin\phi\hat{i} + cos\phi\hat{j}) = -y\hat{i} + (x - a)\hat{j} \tag{16}$$

leading to,

$$\vec{B}_{total} = \frac{\mu_0 I_0 a}{2\pi(R_0^2 - b^2)}\hat{j} \tag{17}$$

8.31 a) We can use the result we have for the magnetic field on the z-axis for a circular current loop. The current in the loop is now,

$$di(r) = \frac{dQ}{dt} = \frac{2\pi r dr\sigma}{T} = \omega r\sigma dr \tag{18}$$

The magnetic field in the z-direction at position z along the disc axis is then,

$$B_z(z) = \int_0^R \frac{\mu_0 di(r) r^2}{2(r^2 + z^2)^{3/2}} = \int_0^R \frac{\mu_0\omega\sigma}{2}\frac{r^3 dr}{(r^2 + z^2)^{3/2}} \tag{19}$$

Using the integral

$$\int drr^3/[(r^2+z^2)^{3/2}] = \frac{r^2+2z^2}{(r^2+z^2)^{1/2}}; \quad yields \quad B_z(z) = \frac{\mu_0\omega\sigma}{2}[\frac{R^2+2z^2}{(R^2+z^2)^{1/2}} - 2z] \tag{20}$$

b) The magnetic moment can be found in two ways. First we can use $dm = idA$, and integrate over r,

$$m = \int_0^R A(r)di = \int_0^R \pi r^2 wr\sigma dr = \frac{\pi\omega\sigma R^4}{4} \tag{21}$$

An alternative approach is to expand the solution to the field to leading order and to compare the result to $\mu_0 m/(2\pi z^3)$, which is the general dipole field formula on the z-axis. The expansion needs to be carried out to second order in $(R/z)^2$, so that,

$$\frac{R^2+2z^2}{(R^2+z^2)^{1/2}} - 2z = 2z\frac{1+R^2/(2z^2)}{(1+(R^2/z^2))^{1/2}} - 2z = 2z(1+\frac{R^2}{2z^2})(1-\frac{1}{2}\frac{R^2}{z^2}+\frac{3}{8}(\frac{R^2}{z^2})^2+...)-2z \tag{22}$$

The leading order term in this expansion is $R^4/4z^3$, so the leading term in the expansion of the field is,

$$B_z(z) \approx \frac{\mu_0\omega\sigma R^4}{8z^3} = \frac{\mu_0 m}{2\pi z^3}; \quad \text{so that} \quad m = \frac{\pi\omega\sigma R^4}{4} \tag{23}$$

which is in agreement with the first method.