PHY481 - Outline of solutions to Homework 8

9.12. The free current is $\vec{j}_f = j_0 \hat{i}$ for $-a \le z \le a$. The magnetic susceptibility is $\chi_m = 0$ inside the slab and χ_m finite outside the slab. Since there is a free current, the first step is to find the magnetic intensity using $\oint \vec{H} \cdot d\vec{l} = i_f$. Since the current is uniform, we expect the field to be uniform. Using the right hand rule, the magnetic intensity is found to be directed along the negative y-axis. The Amperian loop to consider is a rectangle with long side l and short size $2z$ with normal along the x-axis, long sides parallel to the y-axis and centered on the x-axis. We have to consider two cases $z < a, z > a$. For $z < a$ the enclosed free current is $i_f = 2zlj_0$, while for $z > a$ the enclosed free current is $i_f = 2alj_0$. In both cases, the contour integral gives $-2Hl$, so we find that

$$\vec{H} = -zj_0\hat{j}, \quad -a < z < a; \quad \vec{H} = -aj_0\hat{j}, \quad z > a \quad \vec{H} = aj_0\hat{j}, \quad z < -a \tag{1}$$

The magnetic field is then found from $\vec{B} = \mu\vec{H} = \mu_0(1 + \chi_m)\vec{H}$, so that,

$$\vec{B} = -\mu_0 zj_0\hat{j}, \quad -a < z < a; \quad \vec{H} = -\mu_0(1+\chi_m)aj_0\hat{j}, \quad z > a \quad \vec{H} = \mu_0 aj_0\hat{j}, \quad z < -a \tag{2}$$

The magnetization is given by, $\vec{M} = \chi_m\vec{H}$, so that,

$$\vec{M} = 0, \quad -a < z < a; \quad \vec{M} = -\chi_m aj_0\hat{j}, \quad z > a \quad \vec{M} = \chi_m aj_0\hat{j}, \quad z < -a \tag{3}$$

The surface bound currents are then found from $\vec{K}_b = -\vec{M} \wedge \hat{k}$ at $z = a$, and $\vec{K}_b = \vec{M} \wedge \hat{k}$ at $z = -a$. There are no bulk bound currents as $\vec{\nabla} \wedge \vec{M} = 0$ in all regimes where $\vec{M}$ is unifom. The surface bound currents are then,

$$\vec{K}_b = \chi_m aj_0\hat{i}, \quad z = a; \quad \vec{K}_b = \chi_m aj_0\hat{i}, \quad z = -a \tag{4}$$

When the susceptibility is positive, the surface bound currents, K_b, are in the same direction as the free current leading to an enhancement of the field in the material. When the susceptibility is negative the surface bound currents are opposite the free current and reduce the field inside the material.

9.15. Two long thin concentric conducting cylinders of radii a and b are separated by a region of susceptibility χ_m. The inner cylinder carries current $I\hat{k}$, while the outer cylinder carries current $-I\hat{k}$. Using $\oint \vec{H} \cdot d\vec{l} = i_f = 2\pi rH$ for a circular contour, yields,

$$\vec{H} = 0, \quad r < a; \quad \vec{H} = \frac{I}{2\pi r}\hat{\phi} \quad a < r < b; \quad \vec{H} = 0, \quad r > b \tag{5}$$

The magnetization is found from $\vec{M} = \chi_m \vec{H}$, so that,

$$\vec{M} = 0, \quad r < a; \quad \vec{M} = \frac{\chi_m I}{2\pi r}\hat{\phi} \quad a < r < b; \quad \vec{M} = 0, \quad r > b \tag{6}$$

The bound currents are finite only in the regime $a \le r \le b$. There are surface bound currents at $r = a$ and at $r = b$, and we have to check for a bulk bound current in the regime $a < r < b$ as the magnetization is not constant. The surface bound currents are found from

$$\vec{K}_b = \vec{M} \wedge \hat{r} = \frac{\chi_m I}{2\pi b}\hat{k}, \quad \text{at } r = b; \quad \text{and} \quad \vec{K}_b = \vec{M} \wedge \hat{r} = \frac{\chi_m I}{2\pi a}(-\hat{k}) \quad \text{at} \quad r = a \tag{7}$$

The bound currents are thus in opposite directions at a and b. However, when $\chi_m > 0$ both of these bound currents produce a magnetic field inside the material that is larger than that in the absence of the material. Using the expression for the curl in cylindrical co-ordinates (see Table 2.3), the bulk bound currents are given by,

$$\vec{K}_b = \vec{\nabla} \wedge \vec{M} = \frac{1}{r}\frac{\partial(r\frac{I}{2\pi r})}{\partial r} = 0 \tag{8}$$

Thus, even though the magnetization is not uniform the bulk bound current is zero.

9.17. a) Consider an infinite copper cylinder, with susceptibility $\chi_m = -9.6 \times 10^{-6}$, and with field $\vec{B} = B_0\vec{k}$, with $B_0 = 1T$ at the surface of the cylinder. Define $H_0 = B_0/\mu_0$. There are no free currents, so $H_t^{ext} = H_0 = H_t^{int}$. We then have,

$$\vec{B}^{int} = \mu_0(1 + \chi_m)\vec{H}_0; \quad \vec{M} = \chi_m H_0\hat{k}; \quad \vec{K}_b = \vec{M} \wedge \hat{r} = \chi_m H_0 sin\theta\hat{\phi} \tag{9}$$

b) Consider the cylinder with its axis along the z-direction and the field applied along the x-axis. Since there are no free currents, we may take $\vec{H} = -\vec{\nabla}\phi_m$. Gauss's law for magnetism $\vec{\nabla} \cdot \vec{B} = 0$, along with the relation for a linear isotropic material $\vec{B} = \mu\vec{H} = \mu_0(1 + \chi_m)\vec{H}$, implies that $\vec{\nabla} \cdot (\mu\vec{H}) = 0$. In regions of space or in parts of a material where μ is a constant, we have $\mu\vec{\nabla} \cdot \vec{H} = 0$. Using $\vec{H} = \vec{\nabla}\phi_m$ in this equation leads to $\nabla^2\phi_m = 0$. We may thus solve the problem in a manner similar to that used for a dielectric sphere in a uniform electric field, though the boundary conditions are different. The applied field is B_0, which corresponds to $H_0 = B_0/\mu_0$. Assume that the solutions to Laplace's equations correspond to a uniform field in the interior of the cylinder and a uniform field and a dipole in the exterior. Note that this is a dipole in cylinderical co-ordinates (line dipole), so it has a different form than that for point dipoles, we have (the radius of the cylinder is a),

$$\phi_m^{int} = -Ax = -Arcos\phi, \quad r < a; \quad \phi_m^{ext} = -B_0 rcos\phi + C\frac{a^2}{r}cos\phi \quad r > a \tag{10}$$

Now we need two boundary conditions in order to solve for A and C. The first boundary condition comes from the condition $\oint \vec{H} \cdot d\vec{l} = i_f = 0$, which implies that $H_t^{ext} - H_t^{int} = 0$, as there are no free currents. The second condition comes from Gauss's law for magnetism, $\oint \vec{B} \cdot d\vec{A} = 0$, so that $B_n^{ext} - B_n^{int} = 0$. These conditions imply that,

$$\frac{-1}{r}\frac{\partial \phi_m^{ext}}{\partial \phi}|_a = \frac{-1}{r}\frac{\partial \phi_m^{int}}{\partial \phi}|_a; \quad \text{and} \quad -\mu_0 \frac{\partial \phi_m^{ext}}{\partial r}|_a = -\mu \frac{\partial \phi_m^{int}}{\partial r}|_a; \tag{11}$$

which lead to,

$$H_0 = A + C; \quad \mu_0(H_0 + C) = \mu A; \quad \text{solving gives} \quad A = \frac{2H_0}{\chi_m + 2}; \quad C = \frac{\chi_m H_0}{\chi_m + 2} \tag{12}$$

Sometimes this is written in terms of $\kappa_m = \mu/\mu_0 = \chi_m + 1$. The magnetic intensity inside and outside is then,

$$\vec{H}^{int} = \frac{2H_0}{\chi_m + 2}\hat{i}; \quad \vec{H}^{ext} = H_0\hat{i} - \vec{\nabla}(\frac{\chi_m H_0}{\chi_m + 2}\frac{a^2}{r}cos\phi) \tag{13}$$

Using the expression for gradient in cylindrical co-ordinates gives,

$$\vec{\nabla}(\frac{\chi_m H_0}{\chi_m + 2}\frac{a^2}{r}cos\phi) = [\frac{\partial}{\partial r}(\frac{\chi_m H_0}{\chi_m + 2}\frac{a^2}{r}cos\phi)]\hat{r} + [\frac{1}{r}\frac{\partial}{\partial \phi}(\frac{\chi_m H_0}{\chi_m + 2}\frac{a^2}{r}cos\phi)]\hat{\phi}$$

$$= \frac{\chi_m H_0}{\chi_m + 2}\frac{a^2}{r^2}(cos\phi\hat{r} + sin\phi\hat{r}) \tag{14}$$

The dipole term in the external potential, ϕ_m^{ext} is

$$\frac{\chi_m \mu_0 H_0}{\chi_m + 2}\frac{a^2}{r}cos\phi = \frac{\mu_0}{2\pi}\frac{\vec{m}' \cdot \hat{r}}{r} \tag{15}$$

so the line dipole moment (dipole moment per unit length), $\vec{m}'$, is

$$\vec{m}' = \frac{2\pi\chi_m H_0 a^2}{\chi_m + 2} \tag{16}$$

Alternatively we can use

$$\vec{m}' = \vec{M} * \pi a^2 = \chi_m \vec{H}^{int}\pi a^2 = \frac{2\pi\chi_m H_0 a^2}{\chi_m + 2} \tag{17}$$

9.21. The magnetic moment of a current loop is $m = iA$. For a Bohr atom, $mvr = \hbar = m\omega r^2$. The current is

$$i = \frac{q}{T} = \frac{e\omega}{2\pi} = \frac{e\hbar}{2\pi m r^2} \tag{18}$$

The magnetic moment of the Bohr atom is then,

$$m = \pi r^2 i = \frac{e\hbar}{2m} \tag{19}$$

The magnetization is the magnetic moment per unit volume (assuming they are all aligned), so that

$$M = \frac{e\hbar}{2m(3\times10^{-10})^3} \tag{20}$$

10.1 a) The motional emf is $\epsilon = Blv = iR$, so the current is Blv/R
b) The force is $d\vec{F} = id\vec{l}\wedge\vec{B}$, which in this case reduces to a force of magnitude $F = B^2l^2v/R$ in the direction opposite to the velocity.
c) Using Newton's equation, we have,

$$\frac{dv}{dt} = -\frac{B^2l^2v}{mR} = -\alpha R \tag{21}$$

where I defined $\alpha = B^2l^2/Rm$. The solution to this equation, with initial velocity v_0 is, $v(t) = v_0e^{-\alpha t}$. The distance travelled before stopping is

$$x = \int_0^\infty v_0e^{-\alpha t}dt = \frac{v_0}{\alpha} = \frac{Rmv_0}{B^2l^2} \tag{22}$$

d) The power dissipated is i^2R, so the total energy dissipated is

$$U = \int_0^\infty (\frac{B^2l^2v}{R})^2Rdt = \int_0^\infty \frac{B^2l^2}{R}v_0^2e^{-2\alpha t}dt \tag{23}$$

Doing the integral yields $U = mv_0^2/2$, which is the amount of energy at the start.

10.2 The motional emf is $Blv = 480kV$

10.8 The square loop is at distance a from a long straight wire carrying current $I_0cos(\omega t)$. The flux through the loop is given by,

$$\phi = \int \vec{B}\cdot d\vec{A} = \frac{\mu_0i(b-a)}{2\pi}\int_a^b\frac{dr}{r} = \frac{\mu_0i(b-a)ln(b/a)}{2\pi} \tag{24}$$

The current induced in the loop is

$$v_{square} = \frac{d\phi_B}{dt} = \frac{\mu_0I_0\omega sin(\omega t)(b-a)ln(b/a)}{2\pi} \tag{25}$$

and finally the current is v_{square}/R.

10.20 The mutual inductance M in this problem is define through $v_{loop} = Mdi_{wire}/dt$ or $N\phi_{loop} = Mi_{wire}$, with the number of turns in the loop $N = 1$. The equalateral triangle has a flat edge parallel to a wire and distance b from the wire. The apex of the triangle is at distanc $a+b$ from the wire. The flux in the loop is then given by,

$$\phi_{loop} = 2\int_0^l dx \int_0^y \frac{\mu_0 i dy'}{2\pi(b+y')} \tag{26}$$

where $l = a/3^{1/2}$, which is one half the length of a side and $y = xtan60 = 3^{1/2}x$. Evaluating give, $\phi_{loop} = Mi$, with

$$M = \frac{\mu_0 b}{\pi 3^{1/2}}[(1+\frac{a}{b})ln(1+\frac{a}{b}) - \frac{a}{b}] \tag{27}$$

10.27 Energy = volume $\times B^2/(2\mu_0)$.

10.28 The magnetic field of a dipole is,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} \tag{28}$$

and the field energy density is given by, $u = B^2/2\mu_0$, so the total magnetic energy is given by,

$$U = \int u d^3r = 2\pi \int_0^\pi \int_{R_E}^\infty r^2 sin\theta \frac{1}{\mu_0}(\frac{\mu_0}{4\pi r^3})^2 (3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}) \cdot 3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} \tag{29}$$

which gives,

$$U = \int u d^3r = 2\pi \int_0^\pi \int_{R_E}^\infty r^2 sin\theta \frac{1}{\mu_0}(\frac{\mu_0}{4\pi r^3})^2 m^2(3cos^2\theta + 1) \tag{30}$$

or

$$U = \int u d^3r = \frac{\mu_0 m^2}{8\pi^2}\int_0^\pi \int_{R_E}^\infty \frac{1}{r^4} sin\theta(3cos^2\theta+1) = \frac{\mu_0 m^2}{12\pi R_E^3} \tag{31}$$

11.2 a) Plates radius a, separation d with potential difference $V(t)$. Using a circular contour and

$$\int \vec{B}\cdot d\vec{l} = \mu_0 i + \mu_0\epsilon_0 \frac{d\phi_E}{dt} \tag{32}$$

The LHS gives $2\pi B(r)$. The current enclosed is zero. For $r < a$, the electric flux enclosed in the loop is $\phi_E = V\pi r^2/d$, while for $r > a$, the enclosed electric flux is $V\pi a^2/d$. We then find that,

$$B(r) = \frac{\mu_0\epsilon_0 r}{2d}\frac{dV}{dt}; \quad r < a, \quad B(r) = \frac{\mu_0\epsilon_0 a^2}{2rd}\frac{dV}{dt}; \quad r > a \tag{33}$$

b) A wire carrying i has field $\mu_0 i/(2\pi r)$. The current is dQ/dt which is also CdV/dt, where $C = \epsilon_0 A/d = \epsilon_0\pi a^2/d$. Combining these equations, gives the result quoted.

11.13 Maxwell's equations in differential form, in vaccum, are given by,

$$\vec{\nabla}\cdot\vec{E} = 0, \tag{34}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{35}$$

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{36}$$

$$\vec{\nabla} \wedge \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{37}$$

To check that the solutions that are given satisfy these four equations, use Cartesian coordinates. The electric field is given by

$$E_x = 0, \quad E_y = 0, \quad E_z = E_0 cos(\pi x/L) cos(\pi y/L) sin(\omega t) \tag{38}$$

and the divergence of $\vec{E}$ is given by,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{39}$$

The magnetic field is

$$B_x = -B_0 cos(\pi x/L) sin(\pi y/L) cos(\omega t), \quad B_y = B_0 sin(\pi x/L) cos(\pi y/L) cos(\omega t) \quad B_z = 0 \tag{40}$$

so the divergence of $\vec{B}$ is given by,

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + 0 = \frac{B_0 \pi}{L} sin(\pi x/L) sin(\pi y/L) cos(\omega t) - \frac{B_0 \pi}{L} sin(\pi x/L) sin(\pi y/L) cos(\omega t) = 0 \tag{41}$$

The other parts are similar plug and chug

11.21 a) Intensity is the average energy per unit area per unit time. The energy per unit time per unit area or energy flux density is related to the electric and magnetic fields through the Poynting vector

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \wedge \vec{B} \tag{42}$$

The electric and magnetic fields are perpendicular to each other and in phase so the cross product gives

$$Peak\ energy\ flux\ density = \frac{1}{mu_0} E_{peak} B_{peak} \tag{43}$$

The peak energy flux density is twice the intensity, and the peak magnetic and electric fields are related by, $E_{peak} = cB_{peak}$, so we have,

$$1300 W/m^2 = \frac{1}{2\mu_0} c B_0^2 \quad \text{so that} \quad B_0 = 2.3 \times 10^{-6} T; \quad E_0 = 700 V/m \tag{44}$$

Similar calculations for b) and c)

d) $n(t)h\nu = Intensity$ where $n(t)$ is the number of photons arriving per unit area per unit time.