PHY481 - Lecture 10

Sections 3.8, 4.1, 4.2 of PS

A. A dipole in an electric field

The simple view: In the simplest case, we consider a simple dipole consisting of two charges, placed in a uniform electric field. We take the angle between the electric field and the dipole to be θ. Since $\vec{F} = q\vec{E}$ there are equal and opposite forces on the two charges of magnitude q in the dipole, so the net force is zero and the net center of mass motion is zero. However there is a torque on the dipole

$$\vec{N} = \vec{r} \wedge \vec{F} = 2\frac{d}{2} sin\theta qE = \vec{p} \wedge \vec{E} \tag{1}$$

where d is the separation between the two charges in the dipole, and $\vec{p}$ is the dipole moment. The torque is zero when the dipole aligns with the field and $\theta = 0$. The state of zero energy is taken to be at the angle $\theta = 90^o$ where the torque is maximum. The potential energy of the dipole in the field is then found from,

$$U = (-)\int_{\pi/2}^{\theta} pEsin\theta'(-)d\theta' = -pEcos\theta = -\vec{p} \cdot \vec{E} \tag{2}$$

The lowest energy state is when $\theta = 0$ and the dipole is aligned with the applied field. The highest energy state is when $\theta = \pi$ and at this point the torque is also zero, so it is a point of unstable equilibrium. Any slight change in θ away from π makes the dipole have a torque pushing it to towards the lowest energy state.

General calculation: For a general charge distribution, we have,

$$\vec{N} = \int \vec{r} \wedge d\vec{F} = \int \vec{r} \wedge \rho\vec{E}d\vec{r} = \int \rho\vec{r} \wedge \vec{E}d\vec{r} \tag{3}$$

If the field is uniform, then the electric field can be taken out of the integral. We also use the general expression for the dipole $\vec{p} = \int \rho(\vec{r})\vec{r}d\vec{r}$ to find,

$$\vec{N} = \vec{p} \wedge \vec{E} \tag{4}$$

The energy of a dipole in a field can be found by starting with the general expression for the energy cost of placing a small amount of charge a fixed potential,

$$U = \int \rho(\vec{r}')V(\vec{r}')d\vec{r}' = \int \rho(\vec{r}')[V(\vec{r}) + (\vec{r}' - \vec{r}) \cdot \vec{\nabla}V + ...]d\vec{r}' \tag{5}$$

Using $E = -\nabla V$, assuming that the electric field is constant, and taking the total charge $\int \rho(\vec{r}')d\vec{r}' = 0$ to be zero, we find that the last expression on the RHS reduces to,

$$\int \rho(\vec{r}')\vec{r}'d\vec{r}' \cdot \vec{\nabla}V = -\vec{p} \cdot \vec{E} \tag{6}$$

If the total charge is not zero, or if the electric field is not constant, then the dipole will have a center of mass motion in addition to it's rotation toward alignment with the field.

B. Basic properties of conductors

Conductors require care as we cannot simply assume that the charges on the surface of a conductor have fixed locations. Instead the charges move in response to an electric field leading to a coupled problem that in general seems hard to solve. In this chapter we develop some ideas and tools to figure out how a metal responds in the presence of charges and/or an applied electric field.

Things we already know about conductors

(i) In electrostatic equilibrium: Electric field inside a conductor is zero. This implies that the potential inside and on the surface of a conductor is constant and that the net charge inside a conductor is zero. However the charge density on the surface of the conductor can and often is non-zero. In fact this charge density arranges itself to ensure that the electric field inside the conductor is zero.

(ii) The electric field at the surface of a conductor is $\sigma(\vec{r})/\epsilon_0$ and is perpendicular to the surface of the conductor. This is true even if the the surface has a complex shape and if the charge density is different in different parts of the conductor.

One further property that is very useful in solving problems in electrostatics and in many other areas of science and engineering is as follows.

(iii) Uniqueness: If we find a potential function which satisfies the voltage or charge boundary conditions, then it is the unique correct solution. This is a general property of linear partial differential equations with appropriate boundary conditions. This means that if you guess a solution which works, it is the correct solution. This is remarkably useful especially when using superposition to solve problems, as we shall see below.

(iv) When a conductor is grounded, it means that the potential of the conductor is fixed at zero. In general we may specify either the total charge or the potential of a conductor. The total charge may be distributed in complex ways on the surface of the conductor, but

the potential has to be the same everywhere on the surface and in the interior, provided we are at electrostatic equilibrium. In terms of differential equations, when the potential is fixed it is called a Dirichelet boundary condition, while if the charge density is specified it is called a Neuman boundary condition.

Example 1 - a cavity inside a metal

A non-trivial situation is when a cavity is inside a conductor. If there is no net charge in the cavity, then the net charge at the boundary of the cavity remains zero. The only way a net charge occurs on the inner surface is when there is a net charge in the cavity. The proof is trivial. When there is no charge in the cavity, a net charge density on the inner surface would lead to an electric field inside the metal, which is not allowed. In contrast if there is a charge in the cavity a charge must be induced on the inner cavity way to ensure that the field of the charge does not penetrate into the metal. These behaviors are due to the fact that there can be no electric field inside the conductor. Consider a sphere of radius b with a concentric spherical hole of radius r and a point charge q at its center. Consider two cases (i) when the sphere is isolated; (ii) when the sphere is grounded. In both cases find the electric field when $r < a$ and when $r > b$. Do a similar analysis for the analogous cases with cylinders.