PHY481 - Lecture 11

Sections 4.2, 4.3 of PS

A. Electric fields near two charged metal sheets

The two sheets have the same area L^2 and have total charge Q on the top sheet and q on the bottom sheet. The two metal sheets have thickness a and are separated by a distance $d << L$ so we can treat the sheets as infinite. The sheets have normals oriented along the z-axis. Let the charge on the top of the top sheet be Q_{2u} and the charge on the bottom of the top sheet be Q_{2l}. The charge at the top of the bottom sheet is Q_{1u} and the charge at the bottom of the lower sheet is Q_{1l}. We need to find these charges, the electric field between the sheets and the voltage difference between the two sheets. The sheets are isolated so charge conservation requires that,

$$Q_{1u} + Q_{1l} = q; \quad \text{and} \quad Q_{2u} + Q_{2l} = Q \tag{1}$$

In addition we require that the electric field inside the upper metal sheet and inside the lower metal sheet are zero. Using superposition to calculate these electric fields, we find (note that Gauss's law does not help us here),

$$\frac{Q_{2l}}{2\epsilon_0 A} - \frac{Q_{2u}}{2\epsilon_0 A} + \frac{Q_{1l}}{2\epsilon_0 A} + \frac{Q_{1u}}{2\epsilon_0 A} = 0 \tag{2}$$

and

$$-\frac{Q_{2u}}{2\epsilon_0 A} - \frac{Q_{2l}}{2\epsilon_0 A} - \frac{Q_{1u}}{2\epsilon_0 A} + \frac{Q_{1l}}{2\epsilon_0 A} = 0 \tag{3}$$

Combining the second of equations (1) with Eq. (2), we find $Q_{2u} = (Q+q)/2$. Similarly we find that $Q_{2l} = (Q-q)/2$ and $Q_{1u} = -(Q-q)/2$ and $Q_{1l} = (Q+q)/2$. The outer surfaces of the two sheets have the same charge and inner surfaces have charge of the same magnitude by opposite sign. This is consistent with the results found for the case $Q = Q, q = -Q$ where there is no charge on the outer surfaces. Notice that if the charges of the two plates are the same, then all of the charge is on the outer surfaces of the metal sheets.

The electric field is found by superposition or by using Gauss's law. Using Gauss's law is easier, but it is good to check it using superposition. We find,

$$E_{between} = \frac{\sigma}{\epsilon_0} = \frac{-(Q-q)}{2A\epsilon_0}; \quad E_{upper} = -E_{lower} = \frac{(Q+q)}{2A\epsilon_0} \tag{4}$$

Finally the voltage between the plates is $Ed = V$, with the positively charged sheet at the higher potential.

B. Definition of capacitance

Capacitance measures the ability of a system to store charge. It is assumed that if the applied voltage is zero, no (net) charge is stored, and that when a voltage is applied charge starts to be stored. The basic geometry is set up by attaching one piece of metal to the positive lead of a battery, while another piece of metal is attached to the other electrode of a battery. The fundamental relation is then,

$$Q = CV \tag{5}$$

where V is the voltage of the battery and Q is the charge stored on the pieces of metal, $+Q$ on the piece of metal attached to the positive electrode of the battery and $-Q$ on the piece of metal attached to the negative electrode. From the calculation above for two metal sheets we have,

$$Q = \sigma A; \quad E = \frac{\sigma}{\epsilon_0}; \quad \text{and} \quad V = Ed = \frac{Qd}{A\epsilon} \tag{6}$$

Which can be written as,

$$Q = \frac{\epsilon_0 A}{d} V = CV \tag{7}$$

so that the capacitance of a parallel plate capacitor is $C_{plate} = \epsilon_0 A/d$ as you know. However the definition of capacitance of Eq. (11) is true for any linear dielectric, which applies to most materials in the low voltage limit.

Example 1. A coaxial cable

Consider two coaxial cylindrical metal shells of radii a and b, where $b > a$ and $d = b - a$. A voltage V is applied to the outer cylinder and the inner cylinder is grounded so the total charge on the two shells is Q on the outer and $-Q$ on the inner shell. The electric field for $r > b$ and for $r < a$ is zero by Gauss's theorem. The electric field for $a < r < b$ is found from Gauss's law,

$$E 2\pi r L = -Q/\epsilon_0 \quad \text{so that} \quad E(r) = -\frac{Q}{2\pi r L \epsilon_0} \tag{8}$$

Now we find the potential difference by integration,

$$V_b = -\int_a^b \vec{E} d\vec{l} = \int_a^b \frac{Q}{2\pi r L \epsilon_0} dr = \frac{Q}{2\pi L \epsilon_0} log(b/a) \tag{9}$$

The voltage difference is $V = V_b$ as $V_a = 0$ because the inner cylinder is grounded. Note also that we are not using the usual rule that the potential at infinity is zero in this calculation. Anyway all that matters is the potential difference and from Eq. (11)

$$Q = \frac{2\pi L\epsilon_0}{log(b/a)}V, \tag{10}$$

so that $C_{coax} = 2\pi L\epsilon_0/log(b/a)$

Example 2. Concentric spherical shells.

Consider two concentric metal shells of radii a and b, where $b > a$ and $d = b - a$. A voltage V is applied to the outer cylinder and the inner cylinder is grounded so the total charge on the two shells is Q on the outer and $-Q$ on the inner shell. The electric field for $r > b$ and for $r < a$ is zero by Gauss's theorem. The electric field for $a < r < b$ is found from Gauss's law,

$$E4\pi r^2 = -Q/\epsilon_0 \quad \text{so that} \quad E(r) = -\frac{Q}{4\pi\epsilon_0 r^2} \tag{11}$$

Now we find the potential difference by integration,

$$V_b = -\int_a^b \vec{E}d\vec{l} = \int_a^b \frac{Q}{4\pi\epsilon_0 r^2}dr = -\frac{Q}{4\pi\epsilon_0}[\frac{1}{b} - \frac{1}{a}] = \frac{Q(b-a)}{4\pi ab\epsilon_0} \tag{12}$$

Again $V_a = 0$ as the inner cylinder is grounded. We then have,

$$Q = \frac{4\pi ab\epsilon_0}{b-a}V \tag{13}$$

so that $C_{sphere} = 4\pi ab\epsilon_0/(b-a)$

For a general geometry determination of the capacitance requires that we calculated the potential difference between two pieces of metal for a given charge on the metals or equivalently the charge on the two pieces of metal given the potential difference. The latter is easier as the voltage on all parts of the metal is the same and hence we know the voltage boundary conditions for Laplace's equation. Nevertheless in general this problem can only be solved numerically.

C. The method of images

The method of images is a method for reducing some problems where a charge is near a conductor to an equivalent problem involving only point charges with the conductor removed from the problem. The method begins with consideration of a charge near an infinite metal half space. That is for $z \leq 0$ space is occupied by a grounded conductor, while for $z = 0$ it is vaccuum, except for a point charge, q at position z_0. We want to find the the voltage and electric field in the region $z > 0$, and we also want to find the surface charge on the conductor $\sigma(x, y, 0)$.

Lets write down what we know: (i) The voltage for $z \leq 0$ is zero as the conductor is grounded. (ii) The electric field for $z < 0$ is zero at $z = 0$ is in the $\hat{k}$ direction. The general solution obeys $\nabla^2 V = -\rho/\epsilon_0$ and has solution,

$$V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|} \quad \text{or} \quad V(\vec{r}) = \int \frac{k\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \tag{14}$$

Now notice that we can satisfy the condition $V(x, y, 0) = 0$ with the function,

$$V(\vec{r}) = kq[\frac{1}{(x^2 + y^2 + (z - z_0)^2)^{1/2}} - \frac{1}{(x^2 + y^2 + (z + z_0)^2)^{1/2}}] \tag{15}$$

This function has the physical interpretation of a real charge at $z = z_0$ (the first term) and an equal and opposite image charge at $z = -z_0$ (the second term). The basic idea is to combine terms of opposite sign to create a surface where the voltage is zero. This solution is only valid for $z > 0$. For $z < 0$, the solution is $V = 0$. This equation is a solution to Poisson's equation and it satisfies the boundary conditions therefore it is unique and exact. The charge density at the surface of the conductor is found from $\sigma(x, y, 0) = \epsilon_0 E_z(x, y, 0)$ which yields,

$$\sigma(x, y, 0) = \frac{-z_0 q}{2\pi(x^2 + y^2 + z_0^2)^{3/2}} \tag{16}$$

Using superposition, we can solve for the general case of a general charge distribution above a conducting surface at $z = 0$, as,

$$V(\vec{r}) = \int k\rho(\vec{r}')[\frac{1}{|\vec{r}' - \vec{r}|} - \frac{1}{|\vec{r}' - 2z'\hat{k} - \vec{r}|}] \tag{17}$$

which corresponds to making an image charge for every charge in the distribution which lies above $z = 0$. Since for each pair $V = 0$ at $z = 0$, then their sum is also zero. Thus the boundary condition is satisfied and we have the exact, unique solution. A special case is a dipole, which has an equal an opposite image dipole and at long distances we can write down the total potential and field.