PHY481 - Lecture 12

Sections 4.2, 4.3 of PS

A. More on problems solved using image charge methods

First lets solve the simplest image charge problem completely using polar co-ordinates. The geometry is a grounded semiinfinite metal with normal along positive $\hat{k}$ and metal for all $z < 0$. A charge q is placed on the z-axis at position z_0. Find the potential for $z > 0$. From this expression show that the potential at $z = 0$ is zero. Find the electric field and show that at $z = 0$ the electric field is directed along the z-axis. Using the image charge method, we have,

$$V(r,\theta) = kq[\frac{1}{(r^2 + z_0^2 - 2rz_0cos\theta)^{1/2}} - \frac{1}{(r^2 + z_0^2 + 2rz_0cos\theta)^{1/2}}] \tag{1}$$

The surface $z = 0$ corresponds to $\theta = \pi/2$. At this value of θ it is easy to show that the potential. The electric field in polar co-ordinates is given by,

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \tag{2}$$

The $\hat{\phi}$ component is zero as there is no ϕ dependence in the potential. Doing the derivative for the other two components yields,

$$E_\theta = kq[\frac{z_0 sin\theta}{(r^2 + z_0^2 - 2rz_0cos\theta)^{3/2}} + \frac{z_0 sin\theta}{(r^2 + z_0^2 + 2rz_0cos\theta)^{3/2}}] \tag{3}$$

and

$$E_r = kq[\frac{r - z_0cos\theta}{(r^2 + z_0^2 - 2rz_0cos\theta)^{3/2}} - \frac{(r + z_0cos\theta)}{(r^2 + z_0^2 + 2rz_0cos\theta)^{3/2}}] \tag{4}$$

Now we need to show that these equations satisfy the required boundary conditions that $V(z = 0) = 0$ and that $\vec{E}(z = 0) = -E_\theta(z = 0)\hat{n}$, that is the potential at the surface is zero and the electric field is normal to the surface. In polar co-ordinates, the surface is at $V(r, \pi/2)$. Evaluating this demonstrates that $V(r, \pi/2) = 0$ as required. To show that the electric field is normal to the surface we need to show that $E_r(r, \pi/2) = 0$ and $E_\theta(r, \pi/2)$ is finite. These results follow from Eq. (3) and (4), moreover, we find that,

$$E_\theta(r, \pi/2) = \frac{2kqz_0}{(r^2 + z_0^2)^{3/2}} \tag{5}$$

Finally we would like to show that the total induced charge at the surface of the metal is $-q$ to do that we use the fact that at a metal surface, $\vec{E} = \sigma\hat{n}/\epsilon_0$, which follows from Gauss's law. The induced charge at the metal surface is given by,

$$Q = \int \sigma dA = -\int_0^\infty 2\pi r dr \epsilon_0 E_\theta = -\int_0^\infty 2\pi r dr \epsilon_0 \frac{2kqz_0}{(r^2+z_0^2)^{3/2}} = -q \tag{6}$$

The minus sign in the second expression arises as the θ direction is $-\hat{n}$ for this surface.

Some problems can be solved using many image charges, one example where four image charges works is the case of a charge q at $(a,a,0)$ when the regions $y < 0$ and $x < 0$ are conducting and grounded. In that case three image charges $-q$ at $(a,-a,0)$, q at $(-a,-a,0)$ and $-q$ at $(-a,a,0)$ ensure that the potential on the surfaces of the metal regions are all zero. In a similar way, the method of images can be used for a wedge of angle π/n (the case above has wedge angle $\pi/2$) when a charge is placed on the central axis of the wedge. In that case alternating image charges placed at symmetric positions at the centers of all wedges inside the metal sum perfectly to ensure that the potential is zero on the wedge surfaces.

In some cases an infinite set of image charges can be used with one example being a charge lying between two grounded metal sheets at locations $z = d$ and $z = -d$. The solution is found by adding image charges iteratively to find,

$$V(r) = kq[\frac{1}{r} + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(r^2+4d^2k^2)^{1/2}}] \tag{7}$$

This can be generalized to the non-symmetric case by using a similar procedure. This approach can also be used for some problems involving two spheres or two cylinders.

Another case which can be solved using one image charge is the case of a charge outside a grounded spherical conductor. Now we can try to solve the problem in the same way as for the flat surface but now using spherical co-ordinates centered at the sphere center. There is azimuthal symmetry so we can restrict consideration to a function $V(r,\theta$, and we have the boundary condition $V(R,\theta) = 0$ as the sphere is grounded. The point charge is at $z_0 > R$. In this case we need to allow the image charge to have a general location and a general charge so that,

$$V(r,\theta) = k[\frac{q}{(r^2+z_0^2-2rz_0cos\theta)^{1/2}} + \frac{q'}{(r^2+z_0'^2-2rz_0'cos\theta)^{1/2}}] \tag{8}$$

Clearly there are two unknowns which can be determined by choosing two convenient locations, for example $\theta = 0, \pi$, $r = R$. These two cases imply that,

$$\frac{kq}{R-z_0} \pm \frac{kq'}{R-z_0} = 0; \quad \frac{kq}{R+z_0} \pm \frac{kq'}{R+z_0} = 0; \tag{9}$$

Taking the minus sign of the first equation with the positive sign on the second equation and subtracting the two equations yields,

$$q' = \frac{-qR}{z_0}, \quad z_0' = \frac{R^2}{z_0} \tag{10}$$

Using these expressions in Eq. (2) and simplifying yields,

$$V(r,\theta) = kq[\frac{1}{(r^2 + z_0^2 - 2rz_0cos\theta)^{1/2}} - \frac{R}{(r^2z_0^2 + R^4 - 2rz_0R^2cos\theta)^{1/2}}] \tag{11}$$

Now lets check that this is zero for all points on the surface of the sphere, by evaluating at $r = R$, where,

$$V(R,\theta) = kq[\frac{1}{(R^2 + z_0^2 - 2Rz_0cos\theta)^{1/2}} - \frac{R}{(R^2z_0^2 + R^4 - 2z_0R^3cos\theta)^{1/2}}] \tag{12}$$

It is clear that this is zero (take R^2 out of the square root in the second term). The electric field in $\vec{E} = -\vec{\nabla}V = (E_r, E_\theta, E_\phi)$, where $E_\phi = 0$ as there is no ϕ dependence in the potential. For the other two components, we have,

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{kq}{r}[\frac{-rz_0sin\theta}{(r^2 + z_0^2 - 2rz_0cos\theta)^{3/2}} + \frac{rz_0R^3sin\theta}{(r^2z_0^2 + R^4 - 2rz_0R^2cos\theta)^{3/2}}] \tag{13}$$

Evaluating this at $r = R$ it is again easy to see that it reduces to zero (take R^2 out of denominator of the second term leads to a factor of R^3, which cancels with the R^3 in the numerator). This is necessary to ensure that there is no parallel or tangential component to the electric field at the surface of the sphere. ie. the field must be normal to the surface and this normal component is given by the electric field in the radial direction,

$$E_r = -\frac{\partial E}{\partial r} = -kq[-\frac{r - z_0cos\theta}{(r^2 + z_0^2 - 2rz_0cos\theta)^{3/2}} + \frac{rz_0^2R - z_0R^3cos\theta}{(r^2z_0^2 + R^4 - 2rz_0R^2cos\theta)^{3/2}}] \tag{14}$$

Evaluating at $r = R$ gives,

$$E_r(R) = -kq[-\frac{R - z_0cos\theta}{(R^2 + z_0^2 - 2Rz_0cos\theta)^{3/2}} + \frac{R^2z_0^2 - z_0R^3cos\theta}{(R^2z_0^2 + R^4 - 2Rz_0R^2cos\theta)^{3/2}}] \tag{15}$$

which reduces to,

$$E_r(R) = \frac{kq}{R}[\frac{R^2 - z_0^2}{(R^2 + z_0^2 - 2Rz_0cos\theta)^{3/2}}] \tag{16}$$

so the surface charge on the sphere $\sigma(\theta,\phi) = \epsilon_0 E_r(\theta,\phi)$ is given by,

$$\sigma(\theta) = \frac{q(R^2 - z_0^2)}{4\pi R(R^2 + z_0^2 - 2Rz_0cos\theta)^{3/2}} \tag{17}$$

The method of solution given above extended to a large number of other cases of a point charge near a sphere or inside a spherical cavity. We shall discuss a few extensions that can be solved simply by using the solution above.

Consider an isolated sphere with total charge Q_0 (Neuman boundary conditions). We can solve this problem using superposition by placing a charge $Q_0 - q'$ at the origin of the sphere to ensure that the potential on the surface of the sphere is constant. For example, in the case of an uncharged but isolated sphere, we take the ground sphere soluation given above and in addition place a second image charge $Q_0 = -q'$ at the origin. This is an example of a two image charge solution. This simple extension satisfies all the boundary conditions and hence is unique and exact.

If instead we fix the potential of the sphere to a finite value (Dirichelet boundary conditions) ie. $V_0 \neq 0$, then the potential outside the sphere is the same as that given above for the grounded sphere plus the term RV_0/r. This additional term satisfies Laplace's equation and the boundary condition so it is the correct solution. Many other problems involving a point charge and a sphere can be solved in a similar manner. For example a point charge inside a spherical cavity inside a conductor.