PHY481 - Lecture 14

End of Chapter 4 and Start of Chapter 5 of PS

Solutions to Laplace's equations

By now we have gone through several methods for find the electric field and electrostatic potential of charge distributions. The basic tools where Gauss's law and superposition (integration e.g. for rods, discs etc.). In the last few lectures we have extended the superposition method to the case of metals where we found the induced charge on various metal shapes using the method of images. Now we start using the differential equation method, ie. Poisson's equation $\nabla^2 V = -\rho/\epsilon$. Actually in many cases we are interested in the electric field and electrostatic potential in places where there is no charge density, so we need to solve the simpler equation $\nabla^2 V = 0$, which is Laplace's equation. First we solve a simple problem using this approach.

A. Conducting sphere or cylinder in a constant electric field

Now consider the case of a grounded conducting sphere placed in an electric field. In this case it is not clear how to use the method of images as there are no charges defined in the problem. Nevertheless we can use what we know about the solutions to Laplace's equation to solve the problem. We know the form of the multipole expansion and taking the monopole and dipole terms we have $V(r,\theta) = A/r + B + Ccos\theta/r^2$ where the angle θ is with respect the the electric field direction. In addition a constant electric field, E_0, in the $\hat{k}$ direction leads to a potential $-E_0 z = -E_0 rcos\theta$. Collecting these terms by superposition we argue that the potential can have the general form,

$$V(r,\theta) = \frac{A}{r} + B + C\frac{cos\theta}{r^2} + Drcos\theta \tag{1}$$

We know that each term in this expression satisfies Laplaces equation, so the sum does too - that is superposition again. Note that this is not the general form of the solution in spherical co-ordinates, but it will turn out to be sufficient for the problem of a conducting sphere in an electric field. We don't expect the first two terms to contribute as the first corresponds to a net charge on the sphere and the second to a finite potential. By symmetry we expect the first to be zero and the second is zero as the sphere is grounded. To ensure that the solution is correct as $r \to \infty$, we need $D = -E_0$. Now notice that the last two terms have the same angular dependence and cancel on the surface of the sphere if we choose $C = a^3 E_0$, therefore the solution is,

$$V(r,\theta) = -E_0 cos\theta[r - \frac{a^3}{r^2}] \tag{2}$$

It is then easy to show that

$$E_r = E_0(1 + \frac{2a^3}{r^3})cos\theta; \quad E_\theta = -E_0(1 - \frac{a^3}{r^3})sin\theta \tag{3}$$

Notice that $E_\theta(a,\theta) = 0$ as it must be to ensure that the electric field at the surface of the conductor is perpendicular to the surface. The charge density at the surface is $\sigma(\theta) = 3\epsilon_0 E_0 cos\theta$. Integrating over the sphere surface gives the net charge on the sphere,

$$Q = \int_0^\pi 2\pi a^2 sin\theta 3\epsilon_0 E_0 cos\theta d\theta = 0 \tag{4}$$

which shows that no charge is transferred from ground to the sphere. In that case this result also applies when the sphere is isolated. However if we integrate over the top hemisphere,

$$Q = \int_0^{\pi/2} 2\pi a^2 sin\theta 3\epsilon_0 E_0 cos\theta d\theta = 3\pi a^2 \epsilon_0 E_0 \tag{5}$$

we see that a positive charge is induced in the top hemisphere. There is an equal amount of negative charge on the lower hemisphere. In a multipole expansion this system has no monopole term, but it has a dipole term. The dipole strength is proportional to the applied electric field.

If the sphere is charged, with an additional charge Q_0, this additional charge is distributed uniformly on the sphere surface and its effect on the potential is found by superposition (see e.g. Griffiths problem 3.20), i.e. we add kQ_0/r to the potential. If we raise the potential of the sphere to a value V_0, this is equivalent to placing a charge $Q_0 = aV_0/k$ on the sphere.

Exactly the same procedure works for conducting cylindrical in a constant applied electric field. In this case, the form of the potential is,

$$V(r,\phi) = Alnr + B + C\frac{cos\phi}{r} + Drcos\phi \to -E_0 r cos\phi + E_0 \frac{a^2 cos\phi}{r}, \tag{6}$$

by following the same reasoning as for the spherical case above. The electric field components are $E_r = E_0 cos\phi(1 + a^2/r^2)$; $E_\theta = -E_0 sin\phi(1 - a^2/r^2)$ and the surface charge density is $2\epsilon_0 E_0 cos\phi$. Integration shows that there is no net charge transfer to the conducting cylinder so this solution also applies for an isolated conduting cylinder in a constant applied field. Addition of a charge to an isolated cylinder leads to a uniform charge density on the

surface of the cylinder and an additional term $Alnr$ in the potential.
NOTE: this calculation does not allow the cylinder to move. What if we allow the cylinder to rotate? What orientation do you think it will adopt?

B. Proof of uniqueness

Dirichelet V is fixed on the boundaries, Neuman $\partial V/\partial n$ is fixed on the boundaries. Uniqueness theorem states that if we find two solutions to Laplace's equation V_1 and V_2, then V_1 and V_2 are the same for Dirichelet boundary conditions, while V_1 and V_2 differ at most by a constant for Neumann boundary conditions.

Proof We have,

$$\nabla^2 V_1 = \nabla^2 V_2 = 0, \quad \text{so that} \quad \nabla^2(V_1 - V_2) = \nabla^2 v = 0 \tag{7}$$

where we defined $v = V_1 - V_2$. For Dirichelet conditions on the boundary we have $V_1 - V_2 = v0$, while for Neumann conditions on the boundary $\partial(V_1 - V_2)/\partial = \partial v/\partial n = 0$. Apply the divergence theory to the funtion $v\vec{\nabla}v$, so that,

$$\int_{vol} \vec{\nabla} \cdot v\vec{\nabla} v d^3r = \oint_{Sur} v\vec{\nabla}v \cdot d\vec{A} = \oint_{Sur} v\frac{\partial v}{\partial n} dA = 0 \tag{8}$$

Now note that $\vec{\nabla} \cdot v\vec{\nabla}v = v\nabla^2 v + (\vec{\nabla}v)^2$. But $\nabla^2 v = 0$, so we have,

$$\int_{vol} (\vec{\nabla}v)^2 d^3r = \oint_{Sur} v\frac{\partial v}{\partial n} dA = 0 \tag{9}$$

Now the RHS is zero, while the LHS is an integral over a quantity that is zero everywhere, therefore $\vec{\nabla}v = 0$. Therefore $v = constant$. For Dirichelet boundary conditions, we know that $v = 0$ on the boundary, where the constant is specifid to be zero, therefore $V_1 = V_2$. For Neumann boundary conditions, the constant cannot be specified, so V_1 and V_2 can differ by a constant.

C. Separation of variables

Separation of variables is a standard tool in all branches of physics as well as all other branches of science and engineering. We will go through separation of variables for the three co-ordinate systems of interest, namely cartesian, polar and cylindrical co-ordinates. Alternative methods include Fourier transformation, Laplace transformation.... You will see that the monopole, dipole, etc solutions occur in Laplace's equation and that the separation of variables method is a way to find all possible solutions. Laplace's equation can be solved in any dimension.

When variation only occurs in one direction

In one dimensional cases where there is for example a dependence on x but not on y or z the solution only depends on x so we have,

$$\frac{d^2V}{dx^2} = 0; \quad \text{so that} \quad V = a + bx \tag{10}$$

In this one dimensional case, the potential must be linear - this is like the potential near a uniformly charged sheet. The values of a and b are set by the boundary conditions. We have learned nothing new here. ie. it's kinda trivial in one dimension. Note that this is not true when there is time dependence or when we need to solve eigenvalue problems, like in quantum mechanics. However in electrostatics it's much simpler.