

PHY481 - Lecture 15

Chapter 5.1-5.3 of PS - We will not cover 5.4, 5.5

Solutions to Laplace's equations with variations in two directions

In two dimensional cases where there is no dependence on one of the directions (e.g. z) but there is dependence on the other two directions e.g. x, y . In this kind of problem, we can choose to use either Cartesian co-ordinates or cylindrical co-ordinates. The domain on which the solution is required usually determines our decision of which co-ordinate system to use. If we are asked to solve a problem in a square or rectangular domain, then it is usually better to go with Cartesian, while on a circular domain it better to use cylindrical co-ordinates. In practice the domain is usually irregular so numerical computations are required. It is relatively simple to do simulations in Cartesian co-ordinates - As we will see later.

A. Separation of variables in Cartesian co-ordinates in two dimensions

Laplace's equation is then,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (1)$$

This can no longer be directly integrated, so we try to reduce it to a simpler form by assuming a solution of the form $V(x, y) = X(x)Y(y)$. Substituting this expression into Eq. (1), we find that,

$$\frac{d^2 X}{dx^2} Y + X \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \text{constant} = -k^2 \quad (2)$$

The requirement that the last expressions have to equal a constant occurs as if we have $f(x) = g(y)$, then both f and g must be constant. It is convenient to call the constant k^2 as we shall see. Also the sign difference between the X equation and the Y equation, and whether k is real or imaginary, is very important and determines whether the solutions to X or Y are oscillatory or decaying functions. From this expression we have,

$$X(x) = A'(k)e^{ikx} + B'(k)e^{-ikx}; \quad Y(y) = C'(k)e^{ky} + D'(k)e^{-ky} \quad (3)$$

A more convenient form of these solutions is to write them as combinations of odd and even functions, namely,

$$X(x) = A(k)\cos(kx) + B(k)\sin(kx); \quad Y(y) = C(k)\cosh(ky) + D(k)\sinh(ky) \quad (4)$$

The constants A, B, C, D can in general depend on the separation constant k . We can also add the one dimensional solutions ie $ax + b$ and $cy + d$ which correspond to the separation constant $k = 0$. The values of k that are used depends on the problem and often an infinite series of values is required. That is, we use superposition to add solutions with different values of the separation constant k . This gives us a lot of freedom in choosing which terms to try, and a lot of constants that we can use to find a solution - this can lead to confusion. The key clarifying property is again uniqueness. If we find a solution which satisfies the boundary conditions, we need to look no further.

A Simple Example. Consider a case where we have two parallel electrodes separated by distance d with the top electrode with electrostatic potential $V(x, d) = V_0 \sin(\alpha x)$, and the bottom electrode having potential $V(x, -d) = -V_0 \sin(\alpha x)$, ie $k = \alpha$ and all values of $A(k) = 0$, and all values of $B(k) = 0$, except $B(\alpha)$. Therefore we take $X(x) = B(\alpha) \sin(\alpha x)$ which satisfies the x equation, provided $k = \alpha$. We take the y -dependence to be, $C(\alpha) \cosh(\alpha y) + D(\alpha) \sinh(\alpha y)$. Since the boundary conditions are odd in y , we choose the sinh part and hence try as our solution, $V(x, y) = C' \sin(\alpha x) \sinh(\alpha y)$. It is easy to show that to satisfy the boundary conditions, we must have, $C = V_0 / \sinh(\alpha d)$. Therefore the solution to the problem is

$$V(x, y) = V_0 \sin(\alpha x) \sinh(\alpha y) / \sinh(\alpha d) \quad (5)$$

In this case, the solution is easy as the boundary conditions correspond to a single "mode" of the x -solutions. In cases where this does not apply, the solution is more complex, as we shall see below. However before doing that, lets go through the separation of variables solution in circular co-ordinates.