PHY481 - Lecture 16

Chapter 5.1-5.3 of PS - We will not cover 5.4, 5.5

A. Separation of variables in circular co-ordinates

Circular co-ordinates are the (r,ϕ) part of cylindrical polar co-ordinates. The Laplacian in cylindrical polar co-ordinates is given by (see Table 2.3, page 34 of PS),

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1}$$

If there is no dependence on z, the third term is zero, so

$$r\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{\partial^2 V}{\partial \phi^2} = 0 \tag{2}$$

Assuming that $V = R(r)\Phi(\phi)$, we find,

$$\frac{r}{R}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{3}$$

We now separate the equation using,

$$\frac{r}{R}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) = \frac{-1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = k^2 \tag{4}$$

The ϕ equation has solutions $cos(k\phi), sin(k\phi)$ when k is finite. When $k = 0$, it has solution $C + D\phi$. Noting that $\Phi(\phi) = \Phi(\phi + 2\pi)$ is required to ensure that Φ is single valued, we set $k = n$, where $n = 0, 1....$ We could also use n negative, but that would just change the sign of the constants D_n in a general solution of the form $C_n cos(n\phi) + D_n sin(n\phi)$. The r equation has the form,

$$r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0 \tag{5}$$

which for $n \neq 0$ has solutions $R = A_n r^n + B_n r^{-n}$, while for $n = 0$, $R = A + Blnr$. With these observations, we find that the general solution to Laplace's equation in circular co-ordinates (ie. cylindrical co-ordinates with no dependence on z) is, The general solution is,

$$V(r,\phi) = (A + Blnr)(C + D\phi) + \sum_{n=1}^{\infty}(A_n r^n + \frac{B_n}{r^n})(C_n cos(n\phi) + D_n sin(n\phi)) \tag{6}$$

A simple example. Consider an uncharged conducting cylinder of radius R has its axis along the z-axis. A constant electric field, E_0, is applied to the cylinder, with the field oriented along the x-axis. Assume that the cylinder is at potential $V = 0$ and has radius R. Find the electrostatic potential for $r > R$.

Solution. The potential as $r \to \infty$ is $-E_0 x$, which in cylindrical co-ordinates is $-E_0 r cos\phi$. The first thing to do is try the simplest solution which only contains this term at infinity, that is,

$$V(r,\phi) = (A'r + \frac{B'}{r})cos\phi \tag{7}$$

To satisfy the boundary condition at infinity, we can take $A' = -E_0$, giving,

$$V(r,\phi) = (-E_0 r + \frac{B'}{r})cos\phi \tag{8}$$

The boundary condition $V = 0$ at $r = R$ implies that,

$$V(R,\phi) = 0 = (-E_0 R + \frac{B'}{R})cos\phi \tag{9}$$

This is satisfied provided $B' = E_0 R^2$, so we have,

$$V(r,\phi) = (-r + \frac{R^2}{r})E_0 cos\phi \tag{10}$$

The electric field should be normal to the surface of the conductor, so we also need to check that $E_\phi(R,\phi) = 0$. We have,

$$E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = (-1 + \frac{R^2}{r^2})E_0 sin\phi \tag{11}$$

This is clearly zero at the surface of the cylinder where $r = R$, as required. So that's it for this example - again a simple case where only one term in the superposition is required.

B. Separation of variables in spherical polar co-ordinates

The Laplacian in spherical polar co-ordinates is the most important case and is an essential part of quantum mechanics as well as EM. Laplace's equation in spherical polar co-ordinates is,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{12}$$

Again lets start with the case where there is no dependence on ϕ, so $V(r,\theta,\phi) \to V(r,\theta) = R(r)\Theta(\theta)$. Making this substitution into Laplace's equation and dividing through by $R\Theta$ gives,

$$\nabla^2 V = \frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = -\frac{1}{\Theta sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial \Theta}{\partial \theta}) = C = l(l+1) \tag{13}$$

The reason why we choose the separation constant $C = l(l+1)$ will become clear later. With this choice, we have the two equations,

$$\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - l(l+1)R = 0 \tag{14}$$

and

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) + l(l+1)\Theta = 0 \tag{15}$$

The R equation is solved by $R(r) = A(l)r^l + B(l)/r^{l+1}$. The Θ equation is more interesting and requires a series solution. First we make the substitution, $u = cos\theta$ and $P(u) = \Theta(\theta)$, which imply that

$$\frac{\partial}{\partial\theta} = -sin\theta\frac{\partial}{\partial u} \tag{16}$$

so that,

$$\frac{d}{du}[(1-u^2)\frac{\partial P(u)}{\partial u}] + l(l+1)P(u) = 0 \tag{17}$$

This equation is call Legendre's equation and is solved using a series solution, $P(u) = \sum_n C_n u^n$. Substituting this series into Legendre's equation we find that,

$$\frac{\partial}{\partial u}\sum_{n=1}^{\infty}(1-u^2)C_n n u^{n-1} + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{18}$$

and,

$$\sum_{n=2}^{\infty} C_n n(n-1)u^{n-2} - \sum_{n=1}^{\infty} C_n n(n+1)u^n + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{19}$$

The coefficient of u^n in this equation must be zero, implying that,

$$C_{n+2}(n+2)(n+1) + C_n[l(l+1) - n(n+1)] = 0; \quad \text{hence} \quad C_{n+2} = C_n\frac{n(n+1) - l(l+1)}{(n+1)(n+2)} \tag{20}$$

For a given value of l, we get two series of solutions, one starting with a value of C_0 and producing C_2, C_4... and the other starting with C_1 and producing C_3, C_5... The key physical observation is that if l is an integer, the series terminates at $n = l$, leading to a finite polynomial solution. In contrast if l is not an integer, the series does not terminate and the coefficients remain finite at infinity, a solution that does not have physicial meaning. The constants C_0 and C_1 are fixed by requiring that the polynomials be normalized on the interval $[-1, 1]$, which corresponds to $[-\pi, \pi]$ in the original variable θ. The normalization condition is

$$\int_{-1}^{1} P_l^2(u)du = \frac{2}{2l+1} \tag{21}$$

With this choice the first few Legendre polynomials are,

$$P_0(u) = 1; \quad P_1(u) = u; \quad P_2(u) = (3u^2 - 1)/2; \quad P_3(u) = (5u^3 - 3u)/2 \tag{22}$$

Higher order functions $P_l(u)$, are found by choosing an arbitrary starting constant, applying the recursion relation till termination, then normalization using Eq. (21) fixes the starting constant.

The solution to Laplace's equation for cases where there is no dependence on ϕ is then of the form,

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta) \tag{23}$$

If we set $A_l = 0$ in this expansion, we get terms which are of the form of the general multipole expansion, with B_0 the monopole term, B_1 the dipole term and B_2 the quadrupole term. In lecture 9, we carried out the multipole expansion to order $l = 2$ using the expansion,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \frac{1}{r} + \frac{r_i cos\theta_i}{r^2} + \frac{r_i^2}{2r^3}[3cos^2\theta_i - 1] + ... \tag{24}$$

Now we recognize the angle dependent terms in this expansion as Legendre polynomials. In fact this expansion can be carried out to arbitrary order in terms of Legendre polynomials yielding,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \sum_{l=0}^{\infty} \frac{r_i^l}{r^{l+1}} P_l(cos\theta) \quad r > r_i, \tag{25}$$

which of course reproduces the multipole terms we found before. Also since this is an expansion to infinite order, we can extend it to the regime $r < r_i$, where the expansion takes the form,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \sum_{l=0}^{\infty} \frac{r^l}{r_i^{l+1}} P_l(cos\theta) \quad r < r_i, \tag{26}$$

Of course these two terms are just the terms in the general solution to Laplace's equation in polar co-orindates (Eq. ()).