PHY481 - Lecture 17

Chapter 5.1-5.3 of PS - We will not cover 5.4, 5.5

A. Legendre Polynomials - Series solution

$$\frac{\partial}{\partial u}[(1-u^2)\frac{\partial P(u)}{\partial u}] + l(l+1)u = 0 \tag{1}$$

This equation is call Legendre's equation and is solved using a series solution, $P(u) = \sum_{n=0}^{\infty} C_n u^n$. Substituting this series into Legendre's equation we find that,

$$\frac{\partial}{\partial u}[1-u^2]\sum_{n=1}^{\infty} C_n n u^{n-1} + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{2}$$

and,

$$\sum_{n=2}^{\infty} C_n n(n-1)u^{n-2} - \sum_{n=1}^{\infty} C_n n(n+1)u^n + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{3}$$

The coefficient of u^n in this equation must be zero, implying that,

$$C_{n+2}(n+2)(n+1) + C_n[l(l+1) - n(n+1)] = 0; \quad \text{hence} \quad C_{n+2} = C_n\frac{n(n+1)-l(l+1)}{(n+1)(n+2)} \tag{4}$$

For a given value of l, we get two series of solutions, one starting with a value of C_0 and producing C_2, C_4... and the other starting with C_1 and producing C_3, C_5... The key physical observation is that if l is an integer, the series terminate at $n = l$, leading to a finite polynomial solution. In contrast if l is not an integer, the series does not terminate and the coefficients remain finite at infinity, a solution that does not have physicial meaning. The constants C_0 and C_1 are fixed by requiring that the polynomials be normalized on the interval $[-1, 1]$, which corresponds to $[-\pi, \pi]$ in the original variable θ. The normalization condition is

$$\int_{-1}^{1} P_l^2(u)du = \frac{2}{2l+1} \quad \text{or} \quad P_l(1) = 1 \tag{5}$$

With this choice the first few Legendre polynomials are,

$$P_0(u) = 1; \quad P_1(u) = u; \quad P_2(u) = (3u^2-1)/2; \quad P_3(u) = (5u^3-3u)/2 \tag{6}$$

Higher order functions $P_l(u)$, are found by choosing an arbitrary starting constant, applying the recursion relation till termination, then normalization using Eq. (5) fixes the starting constant.

The solution to Laplace's equation for cases where there is no dependence on ϕ is then of the form,

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta) \tag{7}$$

If we set $A_l = 0$ in this expansion, we get terms which are of the form of the general multipole expansion, with B_0 the monopole term, B_1 the dipole term and B_2 the quadrupole term. In lecture 9, we carried out the multipole expansion to order $l = 2$ using the expansion,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \frac{1}{r} + \frac{r_i cos\theta}{r^2} + \frac{r_i^2}{2r^3}[3cos^2\theta - 1] + ... \tag{8}$$

Now we recognize the angle dependent terms in this expansion as Legendre polynomials. In fact this expansion can be carried out to arbitrary order in terms of Legendre polynomials yielding,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \sum_{l=0}^{\infty} \frac{r_i^l}{r^{l+1}} P_l(cos\theta) \quad r > r_i, \tag{9}$$

which of course reproduces the multipole terms we found before. Also since this is an expansion to infinite order, we can extend it to the regime $r < r_i$, where the expansion takes the form,

$$\frac{1}{|\vec{r} - \vec{r}_i|} = \sum_{l=0}^{\infty} \frac{r^l}{r_i^{l+1}} P_l(cos\theta) \quad r < r_i, \tag{10}$$

Of course these two terms are just the terms in the general solution to Laplace's equation in polar co-orindates (Eq. (7)).

A simple example. Consider a sphere in a constant applied electric field $\vec{E} = E_0\hat{k}$. The potential is then $V(z) = -E_0 z = -E_0 r cos\theta$. Again we tried the simplest solution first, which corresponds to $l = 1$, so that,

$$V(r\theta) = (A_1 r + \frac{B_1}{r^2})P_1(cos\theta) = (A_1 r + \frac{B_1}{r^2})cos\theta \tag{11}$$

To satisfy the boundary condition at infinity, we set $A_1 = -E_0$, and to satisfy the boundary condition at the surface of the sphere, we require

$$V(R,\theta) = 0 = (-E_0 R + \frac{B_1}{R^2})cos\theta \quad \text{so that} \quad B_1 = E_0 R^3 \tag{12}$$

yielding the solution,

$$V(r,\theta) = (-E_0 r + \frac{E_0 R^3}{r^2})cos\theta \tag{13}$$

From this expression it is clear that the induced dipole moment can be found by comparing,

$$k\frac{p_{ind} cos\theta}{r^2} = \frac{E_0 R^3}{r^2} cos\theta \;\; \text{so that} \quad p_{ind} = 4\pi\epsilon_0 E_0 R^3 \tag{14}$$

Notice that E_0R^2/k has dimensions of charge.

A more interesting example - *Example 1 of PS.* Consider a square region of space, centered at the origin and with of dimensions $a \times a$. The sides of the square are parallel to the x and y axes. The sides at $y = \pm a/2$ are held at a fixed potential V_0, while the sides at $x = \pm a/2$ are held are grounded, ie $V = 0$ there. Find an expression for the potential everywhere on the interior of the square domain.

Solution The first observation is that the boundary conditions in the boundary conditions in the x direction are symmetric about the origin so we choose functions of the form $X(x)$ $cos(kx)$. Similarly the boundary conditions in the y-direction are symmetric so we choose $Y(y)$ $cosh(ky)$. Since there is dependence on both x and y directions, we expect the one dimensional solutions will not be useful, so we do not include them. We then have,

$$V(x,y) = \sum_k A(k) cos(kx) cosh(ky) \tag{15}$$

At this point we don't know what values of k are needed. We can find a set of values of k by imposing the boundary conditions in the x-direction where $V(a/2,y) = V(-a/2,y) = 0$. These boundary conditions can be satisfied by choosing,

$$cos(ka/2) = 0 \quad \text{or} \quad k = \frac{(2n+1)\pi}{a}, \quad \text{with} \quad n = 0,1,2.... \tag{16}$$

Notice that we don't need to include negative values of n due to the fact that the cosine function is even. The electrostatic potential is then given by

$$V(x,y) = \sum_{n=0}^{\infty} A(n) cos((2n+1)\frac{\pi x}{a}) cosh((2n+1)\frac{\pi y}{a}) \tag{17}$$

Our remaining task is to satisfy the boundary conditions in the y-direction, $V(x, \pm a/2) = V_0$, so we need,

$$V_0 = \sum_{n=0}^{\infty} A(n) cos((2n+1)\frac{\pi x}{a}) cosh((2n+1)\frac{\pi}{2}) = \sum_{n=0}^{\infty} A'(n) cos((2n+1)\frac{\pi x}{a}) \tag{18}$$

where $A'(n) = A(n) cosh((2n+1)\pi/2)$. Our problem reduces to finding the Fourier cosine series for a constant function. Due to the orthogonality of the basis functions, we can extract the coefficient $A'(n)$, by multiplying both sides by $cos[(2n'+1)(\pi x/a)]$ and then integrating both sides over the interval $[-a/2, a/2]$,

$$\int_{-a/2}^{a/2} V_0 cos[(2n'+1)\frac{\pi x}{a}]dx = \sum_{n=0}^{\infty} A'(n) \int_{-a/2}^{a/2} cos((2n+1)\frac{\pi x}{a}) cos((2n'+1)\frac{\pi x}{a}) dx \tag{19}$$

Carrying out the integrals, we find that,

$$\frac{2V_0 a}{(2n'+1)\pi} sin((2n'+1)\frac{\pi}{2}) = \sum_n A'(n)\delta_{nn'} \int_{-a/2}^{a/2} [cos((2n+1)\frac{\pi x}{a})]^2 dx = \frac{aA'(n')}{2} \tag{20}$$

Solving we find that $A'(n) = 4V_0(-1)^n/[(2n+1)\pi]$, so the solution to our problem is,

$$V(x,y) = \sum_{n=0}^{\infty} \frac{4V_0(-1)^n}{(2n+1)\pi} \frac{cos((2n+1)\frac{\pi x}{a})cosh((2n+1)\frac{\pi y}{a})}{cosh((2n+1)\frac{\pi}{2})} \tag{21}$$