PHY481 - Lecture 19

Chapter 6 of PS

Chapter 6 of PS covers the use of Maxwell's equations in dielectric media. First we go through some general relations that apply in dielectric media, before specializing to the case of linear dielectrics. Most of the problems in PS involve linear dielectrics.

A. General Relations

Polarization is the primary new effect and leads to reduction (screening) of electric field, as compared to vacuum. Screening is produced by opposing electric fields due to bound charge which in term originates from the polarization. Bound charge may occur at the surface of a dielectric material in which case it is denoted by σ_b, the surface bound charge density or in the bulk where it is the bulk bound charge density, ρ_b. The relations between these quantities and the polarization are,

$$\sigma_b = \hat{n}\cdot\vec{P}; \qquad \rho_b = -\vec{\nabla}\cdot\vec{P} \tag{1}$$

where $\vec{P}$ is the polarization density, i.e. the polarization per unit volume. The bound charge acts as a source of the electric field so we have,

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_f + \rho_b) \tag{2}$$

where ρ_f is the so-called free charge and is really the excess charge. Now we define the displacement field through $\vec{\nabla}\cdot\vec{D} = \rho_f$, so that,

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\vec{\nabla}\cdot\vec{D} - \vec{\nabla}\cdot\vec{P}); \quad \text{so that} \quad \epsilon_0\vec{E} = \vec{D} - \vec{P}. \tag{3}$$

The displacement field can be found by using Gauss's law, that is,

$$\vec{\nabla}\cdot\vec{D} = \rho_f; \quad \text{implies} \quad \oint \vec{D}\cdot d\vec{A} = q_f \tag{4}$$

There are thus several ways to approach problems in dielectrics. The boundary conditions at interfaces also follow from the equations above, for example continuity of the electrostatic potential. Of course we can also still use $\vec{E} = -\vec{\nabla}V$.

In order to solve dielectric problems we need to be given either $\vec{P}$, ρ_b and ρ_f, and/or, we can be given a relation between $\vec{P}$ and $\vec{E}$, i.e. some function or constitutive relation

$\vec{P}(\vec{E})$. We solve a few problems (e.g. 6.5) where we are given the polarization, in which case direction application of the formula above provides a solution. Below we derive the result for a uniformly polarized sphere. However usually we don't know $\vec{P}$ and we need more information than that contained in the results above in order to solve EM problems in dielectric media. The key additional information is usually in the form of a constitutive law giving a relation between the polarization and the local electric field, i.e. some function $\vec{P}(\vec{E})$. At the molecular level we already noted two relations between the degree of alignment of dipoles as a function of applied electric field. These laws can be extended to the polarization as we discuss later. However at this point we focus on one simple constitutive law, that for linear isotropic dielectric materials where $\vec{P} = \epsilon_0 \chi_e \vec{E}$.

B. Linear isotropic dielectrics where: $\vec{P} = \epsilon_0 \chi_e \vec{E}$, $\vec{D} = \epsilon \vec{E}$, $\kappa = \epsilon/\epsilon_0$

Most of the problems that we solve will take the simplest, but still very important, case of a linear isotropic dielectric where,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}; \tag{5}$$

Using the general relation, $\epsilon_0 \vec{E} = \vec{D} - \vec{P}$ with the above equation, we find that,

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \epsilon_0 \chi_e \vec{E}; \quad \text{so that} \quad \vec{D} = \epsilon \vec{E} \ \text{ with } \quad \epsilon = \epsilon_0 (1 + \chi_e) \tag{6}$$

where χ_e is the dielectric susceptibility and ϵ is the permittivity. Dielectric materials are usually characterized by the ratio $\kappa = \epsilon/\epsilon_0$, the relative permittivity. The relative permittivity of air is close to 1, while that of engineering polymers is around 3. High values of κ are found in perovskite materials such as Strontium Titanate and Barium Titanate.

Solution strategies for problems involving dielectrics

The goal is again to find the electric field and the electrostatic potential from which all of the physics follows. The following two strategies are generally useful.

- (i) Use Gauss's law for $\vec{D}$, then find $\vec{E} = \vec{D}/\epsilon$.
- (ii) In regions where there are no free charges, and where ϵ is uniform, then,

$$\nabla \cdot \vec{D} = 0 = \epsilon \nabla \cdot \vec{E} = -\epsilon \nabla^2 V = 0 \tag{7}$$

That is, Laplace's equation holds. We can then use the methods developed in Chapters 4 and 5 to solve a variety of problems. The key thing to get right is the boundary conditions as will be described in more detail below.

Charges inside dielectrics, use $\vec{\nabla} \cdot \vec{D} = \rho_f, \vec{D} = \epsilon \vec{E}$

Solving problems where there are charges inside dielectric materials proceeds by first solving for $\vec{D}$ using either Poisson's equation or by using Gauss's law. Then find the electric field is using $\vec{D} = \epsilon \vec{E}$. Consider a point charge in a dielectric medium. The field around the point charge is found by using

$$\oint \vec{D} \cdot d\vec{A} = q; \quad \text{so that} \quad \vec{D} = \frac{q}{4\pi r^2}\hat{r}; \quad \text{and hence} \quad \vec{E} = \frac{q}{4\pi \epsilon r^2}\hat{r} \tag{8}$$

The electric field inside the dielectric is *reduced* due to the polarization of the dielectric (dielectric screening). Now consider a point charge at the center of a finite dielectric sphere. The field inside the sphere is given above, but what about the field outside the sphere? Following the same considerations as above, we find that the electric field outside the dielectric sphere is *unaltered* by the dielectric - it is useful to compare this result to a charge at the center of an isolated uncharged metal shell.

Now consider a sheet of charge lying in the x-y plane, with charge density σ, inside a dielectric slab that is centered at the origin and has normal $\hat{k}$. If the thickness of the slab is $2d$, the considerations above indicate that the field inside the slab is $\sigma/(2\epsilon)$ if $d > z > 0$. However outside the slab (e.g. for $z > d$ the field is $\sigma/(2\epsilon_0)$ - i.e. the dielectric medium has no effect on the far field.