

PHY481 - Lecture 2

Sections 2.1 and 2.2 of Pollack and Stump (PS)

A. Co-ordinate systems we will use We shall be using three orthogonal co-ordinate systems, cartesian, cylindrical and spherical polar. We need the transformations between these systems. We have,

1. Cartesian co-ordinates

$$\vec{x} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k} \quad (1)$$

2. Cylindrical co-ordinates

$$\vec{x} = (r, \theta, z) = r\hat{r} + z\hat{k} \quad (2)$$

$$x = r\cos\theta; \quad y = r\sin\theta; \quad x^2 + y^2 = r^2 \quad (3)$$

3. Spherical polar co-ordinates

$$\vec{x} = (r, \theta, \phi) = r\hat{r} \quad (4)$$

$$x = r\cos\phi\sin\theta; \quad y = r\sin\phi\sin\theta; \quad z = r\cos\theta; \quad x^2 + y^2 + z^2 = r^2 \quad (5)$$

Following convention and as used in PS r and θ have different meanings for cylindrical as opposed to the spherical polar case.

B. Rotation matrices

Rotation matrices are used to rotate a co-ordinate system about an axis. We may rotate about the x, y, or z axes and there is a different matrix for each case. However the matrices are very similar so we only need to consider one in detail. Lets consider rotation about the z axis. Consider that in the original co-ordinate system the unit vectors are $\hat{i}, \hat{j}, \hat{k}$. We then rotate the co-ordinate system through an angle θ about the z-axis. In this new rotated (primed) co-ordinate system, the new unit vectors along the x', y', z' directions are $\hat{i}', \hat{j}', \hat{k}'$. A vector \vec{x} may be written in either of these co-ordinate systems, ie,

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k} = x'\hat{i}' + y'\hat{j}' + z'\hat{k}' \quad (6)$$

Now it is easy to show that the relationships between the unit vectors in the original and rotated co-ordinate systems are,

$$\hat{i}' = \cos\theta\hat{i} + \sin\theta\hat{j}; \quad \hat{j}' = -\sin\theta\hat{i} + \cos\theta\hat{j}; \quad \hat{k}' = \hat{k} \quad (7)$$

Substituting these expressions in the last of Eq. (6), we find that,

$$\vec{x} = (x'\cos\theta - y'\sin\theta)\hat{i} + (x'\sin\theta + y'\cos\theta)\hat{j} + z'\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} \quad (8)$$

so that,

$$x = x'\cos\theta - y'\sin\theta; \quad y = x'\sin\theta + y'\cos\theta; \quad z = z' \quad (9)$$

or

$$x' = x\cos\theta + y\sin\theta; \quad y' = -x\sin\theta + y\cos\theta; \quad z' = z \quad (10)$$

The latter equation may be written in matrix form,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \vec{x}' = \overline{R}\vec{x} \quad (11)$$

where \overline{R} is the rotation matrix in the middle equation.

It is easy to show that the inverse of \overline{R} is the transpose of \overline{R} , so that,

$$\overline{R}^T \overline{R} = I \quad (12)$$

where I is the identity matrix. Rotation matrices for co-ordinate rotations by angle θ around the x,y,z axis are respectively,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)$$

C. What is a vector? What is an invariant?

A vector is a quantity that behaves like the position vector \vec{x} under co-ordinate rotations. An invariant is unaltered under rotations of the co-ordinate system.

For example, we expect the dot product of two vectors and the cross product of two vectors to be unaltered by rotations of the co-ordinate system as they depend on the angle between the two vectors and not on the angle itself - they should be invariant. E.g. to check

this for the dot product, we need to prove that $\overline{R}\vec{A} \cdot \overline{R}\vec{B} = \vec{A} \cdot \vec{B}$, which is easily seen as follows;

$$\overline{R}\vec{A} \cdot \overline{R}\vec{B} = \vec{A}^T \overline{R}^T \overline{R}\vec{B} = \vec{A} \cdot \vec{B} \quad (14)$$

D. Some help with proving vector identities: Levi-Civita tensor

The completely antisymmetric Levi-Civita tensor, ε_{ijk} , seems obscure, however it helps a lot in proving vector identities and is worth learning. The Levi-Civita tensor has 27 components since i, j, k can each take on three values which refer to co-ordinates in 3-space. However all of its entries are zero except the ones where ijk are all different. There are thus 6 non-zero entries which correspond to the 3! permutations of 1, 2, 3. The values of the 6 non-zero entries are either 1 or -1 depending on the number of transpositions required to take the permutation from the natural ordering 123 to the desired permutation, thus,

$$\varepsilon_{123} = 1; \quad \varepsilon_{132} = -1; \quad \varepsilon_{312} = 1; \quad \varepsilon_{213} = -1; \quad \varepsilon_{231} = 1; \quad \varepsilon_{321} = -1 \quad (15)$$

The cross product may then be written,

$$(\vec{A} \wedge \vec{B})_i = \sum_j \sum_k \varepsilon_{ijk} A_j B_k \quad \text{or in suffix notation} = \varepsilon_{ijk} A_j B_k \quad (16)$$

In suffix or Einstein notation, if an index is repeated it implies that a sum over that index should be carried out.

Example 1. Prove the vector identity $\vec{A} \cdot (\vec{B} \wedge \vec{C}) = (\vec{A} \wedge \vec{B}) \cdot \vec{C}$.

Proof. $\vec{A} \cdot (\vec{B} \wedge \vec{C}) = \varepsilon_{ijk} A_i B_j C_k$. Also $(\vec{A} \wedge \vec{B}) \cdot \vec{C} = \varepsilon_{ijk} A_j B_k C_i$. The latter expression would be equivalent to the former if we were to make the replacement $\varepsilon_{ijk} \rightarrow \varepsilon_{jki}$. This is true as it involves two transpositions of the indices and hence preserves all entries in the Levi-Civita tensor. Note that this is considerably simpler than writing out the two expressions in the identity explicitly.

Example 2. Prove the vector identity $\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Proof. The LHS may be written as

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_i = \varepsilon_{ijk} A_j (\vec{B} \wedge \vec{C})_k.$$

Using the expansion of the cross product again, this becomes,

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_i = \varepsilon_{ijk} \varepsilon_{klm} A_j B_l C_m$$

Now use the identity (you prove this in the first assignment)

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

to find,

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_i = \delta_{il}\delta_{jm}A_jB_lC_m - \delta_{im}\delta_{jl}A_jB_lC_m$$

which reduces to

$$(\vec{A} \wedge (\vec{B} \wedge \vec{C}))_i = A_jB_lC_j - A_jB_jC_i = RHS$$

E. Differential operators - Div, Grad, Curl, Laplacian

The Gradient

The gradient is a vector operator which in Cartesian co-ordinates is given by,

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (17)$$

The gradient of a scalar function (f) is then,

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \quad (18)$$

A co-ordinate independent representation of $\vec{\nabla} f$ is through the definition,

$$df = f(\vec{x} + d\vec{x}) - f(\vec{x}) = d\vec{x} \cdot \vec{\nabla} f \quad (19)$$

In Cartesian co-ordinates we have

$$df = f(\vec{x} + d\vec{x}) - f(\vec{x}) = dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y} + dz \frac{\partial f}{\partial z} \quad (20)$$

which reproduces the definition (19). However the simple expression $df = d\vec{x} \cdot \vec{\nabla} f$ contains a powerful result that we use all the time in electrostatics. To see this result, consider surfaces on which the scalar function is a constant. In electrostatics these surfaces are often equipotentials, surfaces on which the electrostatic potential is a constant. In gravity they are surfaces of constant gravitational potential. If we consider a displacement $d\vec{x}$ along an equipotential or surface of constant f , then clearly $df = 0$ by construction. Since the LHS of Eq. (20) is zero, then RHS must also be zero. Since $d\vec{x}$ is a finite displacement, the only way for the RHS to be zero is for $\vec{\nabla} f$ to be perpendicular to the surface of constant f . In electrostatics this is the same as the statement that the electric field is perpendicular to equipotential surfaces.