PHY481 - Lecture 20

Chapter 6 of PS

A. Planar capacitor containing a dielectric (Examples 3,4,5 of PS).

Consider a parallel plate capacitor of area A and plate separation d that has dielectric material with dielectric permittivity ϵ between the places. (i) Find the electric field between the plates when a charge Q is placed on the capacitor. Find the voltage across the capacitor and the energy stored in the capacitor. (ii) Now consider connecting a battery to the capacitor, with voltage V on the top plate and the lower plate at ground. Find the charge on the capacitor plates and the energy stored in the capacitor. In both cases compare your results to the case where there is no dielectric between the plates. That is, in the two cases of fixed charge and fixed voltage, does the stored energy go up or down when the dielectric is added? Also, if the dielectric slab is free to move, is it drawn into the space between the capacitor plates or is it pushed out. Find the force in both cases (i) and (ii). (This is a typical electro-mechanical actuator, used in doorbells amongst other things).

Solution to (i) When a charge Q is placed on the capacitor, we can use Gauss's law (or superposition), to find the diplacement field,

$$\vec{D} = -\sigma\hat{k}; \quad \text{so that} \quad \vec{E} = -\frac{\sigma}{\epsilon}\hat{k} \tag{1}$$

The voltage across the plates is given by $V = |E|d = \frac{\sigma d}{\epsilon}$. The capacitance is found from $Q = CV$, and using,

$$V = \frac{\sigma d}{\epsilon} = \frac{Qd}{A\epsilon} \quad \text{so that} \quad C = \frac{\epsilon A}{d} \tag{2}$$

and the energy stored in the capacitor is $U = Q^2/2C$. If there is no dielectric between the plates, the capacitance is $C_0 = \epsilon_0 A/d$, and hence,

$$\frac{U}{U_0} = (\frac{Q^2}{2C})/(\frac{Q^2}{2C_0}) = \frac{C_0}{C} = 1/\kappa \quad \text{for} \quad \textit{isolated capacitor} \tag{3}$$

so that the energy is reduced when the dielectric is between the plates. Note that from this calculation we deduce that the energy density by writing,

$$U = \frac{Q^2}{2C} = \frac{\sigma^2 A^2 d}{2\epsilon A} = uV \quad \textit{with} \quad u = \frac{1}{2}\epsilon_0\kappa E^2 \tag{4}$$

where u is the energy density, $V = Ad$ is volume and we used $|E| = \sigma/\epsilon$. Now note that,

$$u = \frac{1}{2}\epsilon_0\kappa E^2 = \frac{1}{2}\vec{D}\cdot\vec{E}. \tag{5}$$

The result $u = \frac{1}{2}\vec{D}\cdot\vec{E}$ is a general result for linear isotropic dielectrics.

Solution to (ii) Now consider the case where the voltage across the plates is fixed to V. In that case

$$U/U_0 = \frac{\frac{1}{2}CV^2}{\frac{1}{2}C_0V^2} = C/C_0 = \kappa \quad \text{for} \quad fixed\ voltage\ across\ capacitor \tag{6}$$

The case of a fixed voltage applies to energy storage, as capacitors are usually charged at fixed voltage, e.g. from a generator or solar cell. High energy storage materials require high values of κ (highly polarizable material) and high V (very good insulator with low leakage).

B. An electric field applied to a uniform dielectric sphere

We consider an uncharged uniform dielectric sphere of radius a and dielectric constant ϵ, in a constant applied field

$$\vec{E}_0 = E_0\hat{k}, \quad \text{so that} \ \ V = -E_0 z = rcos\theta. \tag{7}$$

Since the dielectric sphere is uniform and there is no free charge, we have,

$$\vec{\nabla}\cdot\vec{D} = 0 = \vec{\nabla}\cdot(\epsilon\vec{E}) = \epsilon\vec{\nabla}\cdot\vec{E} = 0. \quad uniform\ \epsilon \tag{8}$$

Using $\vec{E} = -\vec{\nabla}V$ we then find that V still obeys Laplace's equation, so we try the solutions,

$$V_{int} = -C_1 rcos\theta, \quad V_{ext} = -E_0 rcos\theta + \frac{C_2 a^3 cos\theta}{r^2} \tag{9}$$

We impose continuity of V, and the condition $\epsilon_0 E_n^{ext}(a,\theta) = \epsilon E_n^{int}(a,\theta)$, to find,

$$-E_0 + C_2 = -C_1; \quad -E_0 - 2C_2 = -\frac{\epsilon}{\epsilon_0}C_1 \tag{10}$$

which lead to,

$$C_1 = E_0\frac{3}{\kappa+2}; \quad C_2 = E_0\frac{\kappa-1}{\kappa+2} \tag{11}$$

where $\kappa = \epsilon/\epsilon_0$. Using the fact that $V_{int} = -C_1 z$, we find that the electric field inside the sphere is uniform

$$\vec{E}_{int} = -\frac{\partial V_{int}}{\partial z} = E_0\frac{3}{\kappa+2}\hat{k} \tag{12}$$

and that the dipole of the sphere is

$$\vec{p}_{sphere} = \frac{C_2 a^3}{k} = 4\pi\epsilon_0 E_0 a^3\frac{\kappa-1}{\kappa+2} \tag{13}$$

The polarization density is then,

$$\vec{P} = \frac{\vec{p}_{sphere}}{volume} = \frac{\epsilon_0 E_0}{3} \frac{\kappa - 1}{\kappa + 2} \tag{14}$$

Again it is useful to compare this to the limit of a metal where $\kappa \to \infty$.

C. A point charge above a dielectric half-space

The image charge method extends quite straightforwardly to a a point charge q at position d above a uniform dielectric in the lower half-space ($z < 0$). We assume that the electrostatic potential in the upper half space $z > 0$ is given by,

$$V_{above} = \frac{kq}{(x^2 + y^2 + (z-d)^2)^{1/2}} + \frac{kq'}{(x^2 + y^2 + (z+d)^2)^{1/2}} \tag{15}$$

The potential in the lower half-space is given by,

$$V_{below} = \frac{kq''}{(x^2 + y^2 + (z-d)^2)^{1/2}} \tag{16}$$

Continuity of the potential at $z = 0$ and the requirement that D_n be continuous, also at $z = 0$, lead to,

$$q'' = q + q'; \quad (q - q')d = -\kappa q'' d \tag{17}$$

Solving gives,

$$q' = q\frac{1-\kappa}{1+\kappa}; \quad q'' = q\frac{2}{1+\kappa} \tag{18}$$

The electric field inside the dielectric ($z < 0$)is thus like that of a point charge at the same position as the original charge, but with a reduced magnitude, q''.