PHY481 - Lecture 21

Chapter 6 of PS

A. Force pulling a dielectric slab between capacitor plates

To find the force on the slab, we have to solve the problem when the slab partially penetrates the slab. It is easiest to see the effect in the case of an isolated (fixed charge) systems. Consider an $l \times w$ slab where the plates are penetrated by an area $x \times w$ of the slab, while an area $(l-x) \times w$ remains filled with air. The voltage on the plates are at equipotentials (they are metal), so the electric field in both regions is the same. However the capacitance in the two regions is different. The total capacitance can be found by adding the charge on the two parts of the plate,

$$\epsilon E x w + \epsilon_0 E(l-x)w = Q \tag{1}$$

Using $V = Ed$, we then find,

$$C = \frac{Q}{V} = \frac{\epsilon x w}{d} + \frac{\epsilon_0 (l-x)w}{d}. \tag{2}$$

Note that we could have used the law of addition of capacitors to get this result. Now we can find the force on the slab by using

$$\frac{F}{U_0} = -\frac{dU(x)}{dx} = -\frac{d}{dx}\frac{Q^2 d}{2(\epsilon x w + \epsilon_0(l-x)w)} = \frac{Q^2 d w(\epsilon - \epsilon_0)}{2(\epsilon x w + \epsilon_0 (l-x)w)^2} \tag{3}$$

In the case of a fixed voltage system, we have to take into account the work done by the battery or generator.

$$dU = \frac{1}{2} dCV^2 = (dQ)V - Fdx = dCV^2 - Fdx \tag{4}$$

so that,

$$F = \frac{1}{2}\frac{dC}{dx}V^2 = V^2 w \frac{\epsilon - \epsilon_0}{2d} \tag{5}$$

B. Uniformly polarized sphere ($\rho_f = 0$, no applied field)

If we have a sphere where there is a uniform density of polarization, $\vec{P} = P_0 \hat{k}$, then at the surface of a sphere, the bound charge is

$$\sigma_b = \hat{r} \cdot P_0 \hat{k} = P_0 cos\theta. \tag{6}$$

We can construct an electrostatic potential corresponding to this charge distribution, by trying the $l = 1$ spherical solution, in which case,

$$V_{int}(r,\theta) = C_1 r cos\theta; \quad V_{ext}(r,\theta) = \frac{C_2}{r^2} cos\theta \tag{7}$$

The potential must be continuous across at $r = a$, which implies that $C_1 = C_2/a^3$. Using $E_r = -\partial V/\partial r$, the radial component of the electric field obeys,

$$E_r^{ext}(a,\theta) - E_r^{int}(a,\theta) = \frac{\sigma_b}{\epsilon_0} = \frac{2C_2}{a^3}cos\theta + C_1 cos\theta = \frac{2C_2}{a^3}cos\theta + \frac{C_2}{a^3}cos\theta = \frac{P_0 cos\theta}{\epsilon_0} \tag{8}$$

From this we find, $C_2 = P_0 a^3/3\epsilon_0$, $C_1 = P_0/3\epsilon_0$ so that,

$$V_{int} = \frac{P_0 r}{3\epsilon_0} cos\theta = \frac{P_0 z}{3\epsilon_0}; \quad V_{ext} = \frac{P_0 a^3}{3\epsilon_0 r^2} cos\theta \tag{9}$$

The electric field inside the sphere is given by,

$$\vec{E}_{int} = -\frac{\partial V_{int}}{\partial z}\hat{k} = \frac{-P_0 \hat{k}}{3\epsilon_0} \tag{10}$$

The electric field is thus uniform and is in the opposite direction to the polarization. The potential outside the sphere is like that of a dipole, and the dipole moment is found by equating the general expression for a dipole with the potential V_{ext},

$$\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{P_0 a^3}{3\epsilon_0 r^2} cos\theta \tag{11}$$

which shows that the magnitude of the dipole moment of the uniformly polarized sphere is given by, $p = \frac{4\pi}{3} a^3 P_0$. As expected, the magnitude of the dipole moment of the sphere is the volume of the sphere times the uniform polarization of the sphere.

C. Relation between atomic dipoles and polarization

At the beginning of our discussion we noted that there is often a linear relation between the polarization of an atom or molecule and the electric field that is applied to it, $\vec{p} = \alpha\vec{E}$. Now we want to connect this behavior with the behavior of the polarization density used in Maxwell's equations $\vec{P} = \epsilon_0\chi_e\vec{E}$.

Since $\vec{p}$ is for one atom or molecule while $\vec{P}$ is the polarization per unit volume, a naive assumption might be to write $\vec{P} = n\vec{p}$ where n is the number of atoms or molecules per unit volume in the dielectric material. This assumption is valid if the electric field is not altered by the dielectric material, which from the discussion above is rarely true. Nevertheless it is true in very dilute gases. However more generally we need to calculate the electric field that is actually seen by the atom or molecule, which is reduced due to macroscopic polarization of the system. In the Claussius Mossotti approximation, we assume that the local electric field is like that inside a uniformly polarized sphere that is, $\vec{P}/3\epsilon_0$. The electric field actually seen by the atom or molecule is then $\vec{E} + \vec{P}/3\epsilon_0$, so that,

$$\vec{P} = n\alpha\vec{E}_{total} = n\alpha(\vec{E} + \vec{P}/3\epsilon_0) \tag{12}$$

We then find that,

$$\vec{P} = \frac{n\alpha}{(1 - n\alpha/(3\epsilon_0))}\vec{E}; \quad \text{and} \chi_e = \frac{n\alpha/\epsilon_0}{(1 - n\alpha/(3\epsilon_0))} \tag{13}$$

This equation may be inverted to find a relation between the atomic/molecular polarizability and the measured dielectric response,

$$\alpha = \frac{\epsilon_0}{n}\frac{\chi_e}{(1 + \chi_e/3)} = \frac{3\epsilon_0}{n}\frac{\kappa - 1}{\kappa + 2} \tag{14}$$

Notice that the dielectric response is linear, however the dielectric response is singular when the denominator goes to zero.

D. A simple mechanical model for dielectric response at the atomic scale

A simple mechanical model for dielectric polarization at the atomic scale is to consider two charges $q, -q$ connected by a spring of spring constant K. When a uniform electric field is applied to this system, the charges find an equilibrium separation where the electrostatic force balances the spring force,

$$qE = Kd \quad \text{so that} \quad d = qE_0/K \tag{15}$$

The polarization of a dilute gas of these

$$P = npE_0 = nqd = nq^2E_0/K; \quad \text{and} \quad D = \epsilon_0 E + P = \epsilon_0(1 + \frac{nq^2}{\epsilon_0 K})E_0 \tag{16}$$

The energy density, is then

$$u = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{\epsilon_0}{2}(1 + \frac{nq^2}{\epsilon_0 K})E_0^2 = u_{field} + u_{springs} \tag{17}$$