PHY481 - Lecture 23

Chapter 8 of PS, Chapter 5 of Griffiths

A. A circular current ring and magnetic dipoles

Consider a circular loop of radius r centered at the origin and lying in the x-y plane. A steady current, i, flows in the positive $\hat{\phi}$ direction. Find the magnetic field on the z- axis. We use the Biot-Savart where $d\vec{l} = rd\phi\hat{\phi}$ and $\hat{R}$ is along the vector from a position on the circle to a point on the z-axis. If we take the angle to the z axis to be α, from the geometry we find that $d\vec{l} = rd\phi\hat{\phi}$ is perpendicular to $\hat{R}$. The vector $d\vec{l} \wedge \hat{R}$ makes an angle of $90 - \alpha$ to the z-axis and its projection onto the x-y plane is at angle ϕ to the x-axis. On the z-axis, by symmetry, the magnetic field points in the z-direction, and we find,

$$B_z(z) = \frac{\mu_0 i}{4\pi} \int_0^{2\pi} \frac{sin(\alpha) r d\phi}{r^2 + z^2} = \frac{\mu_0 i}{2} \frac{r^2}{(r^2+z^2)^{3/2}} \tag{1}$$

Expanding the expression above at large distances, z, gives,

$$B_z(z) \approx \frac{\mu_0 i r^2}{2z^3}(1 - \frac{3}{2}\frac{r^2}{z^2}) \approx \frac{\mu_0 i A}{2\pi z^3} \tag{2}$$

The leading order behavior is thus like that of a electrostatic dipole - there is no (monopole) term like $1/r^2$ for a localized current loop. However if we consider a point magnetic charge, which we call N, and an opposite point magnetic charge, that we call S. Further imagine that this magnetic dipole where centered at the origin, with N and S separated by distance d. What would the magnetic field look like? First we have to decide what the field due to a magnetic monopole looks like. Though the magnetic monopole has never been found, it is assumed that if it existed, its magnetic field would be just like that of an electric charge, as we would have $\int \vec{B} \cdot d\vec{A} = \mu_0 q_m$, so that,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q_m \hat{r}}{r^2} \tag{3}$$

In that case, the magnetic field of a dipole would be,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \hat{m}}{r^3} \tag{4}$$

Now note that on the $z - axis$, this reduces to,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{2\hat{m}}{r^3} \quad \text{z} - \text{axis}. \tag{5}$$

Comparing this expression with that for the current loop, we see that they are the same, provide we take the magnetic moment of the dipole to be, $\vec{m} = iA\hat{k}$. This is a general result for current loops and provides a general concept that connects current loops to magnetic dipoles. Notice that the origin of the difference between magnetic and electric fields is purely due to the difference in the sources of the fields (charges for the $\vec{E}$ field, current for the $\vec{B}$ field). The fields themselves are perfectly analogous.

Since the magnetic and electric fields are completely analogous, we dipole formulae also apply, so that

$$U = -\vec{m} \cdot \vec{B}; \qquad \vec{\tau} = \vec{m} \wedge \vec{B}. \tag{6}$$

The Lorentz Force law

Lorentz derived a general expression for the force on a charge, q, moving with velocity $\vec{v}$ in electric and magnetic fields $\vec{E}$ and $\vec{B}$,

$$\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B}) \tag{7}$$

A charge moving in a magnetic field

A charge q moving in a magnetic field $\vec{B}$ with velocity $\vec{v}$ experiences a force,

$$\vec{F}_B = q\vec{v} \wedge \vec{B} \tag{8}$$

Or equivalently,

$$F_B = qvBsin(\theta) \tag{9}$$

where θ is the angle between the velocity vector and the magnetic field vector. The direction of $\vec{F}_B$ is given by the right hand rule. Note the following:

(i) The force on the charged particle is always *perpendicular* to both the velocity vector and to the magnetic field vector.

(ii) If the particle moves in the direction of the magnetic field, it experiences no magnetic force.

(iii) If the particle moves perpendicular to the magnetic field it experiences the maximum force.

(iv) Since the force is perpendicular to the magnetic field lines and to the velocity vector, the particle "spirals" around the magnetic field lines. The larger the magnetic field the larger the magnetic force and the tighter the spiral.

(v) No work is done by the magnetic field on the charged particle. The kinetic energy of the particle is therefore a constant, if no other forces act on the charged particle.

Constant $\vec{B}$ field perpendicular to $\vec{v}$

A useful notation for drawing vectors in three dimensions:

- A cross indicates a vector into the page
- A dot indicates a vector coming out of the page

Consider the simplest case in which a particle of charge q has velocity $\vec{v}$ which is perpendicular to the direction of the magnetic field, then,

$$F_B = qvB \tag{10}$$

The direction of this force is perpendicular to both the direction of the magnetic field and the direction of the velocity. It also has constant magnitude. The charged particle then undergoes circular motion, with,

$$m\frac{v^2}{R} = qvB \tag{11}$$

This is an incredibly important phenomenon. A vast array of devices use the ability of magnetic fields to bend charges. Note however that a more precise theory finds that power is dissipated by an accelerating charge, including charges in uniform circular motion. This is why high energy particle accelerators have very large circumference. The radiation from an accelerated charged particle is proportional to the acceleration squared. An important use of this effect is for producing high intensity x-rays for a variety of purposes. Now look at various quantities that are important in the circular motion of a charge in a magnetic field:

(i) Radius of the orbit - $R = mv/qB$.

(ii) Period of the orbit - $\tau = 2\pi R/v = 2\pi m/qB$.

(iii) Frequency (Cylotron frequency) of the orbit $f = qB/2\pi m$.

Notice that the period and frequency of the orbit do not depend on the radius. They only depend on the charge to mass ratio q/m and the applied magnetic field B. In addition, if we know the speed of a charged particle and the magnetic field, we can find the charge to mass ratio. This is used in mass spectrometers. Often the frequency of this orbit is called the cyclotron frequency.

The circular motion described about is the basis of understanding more complex motion. For example

(i) If a charged particle has a component of its velocity parallel to the magnetic field lines, then it spirals around the magnetic field lines in a helical motion.

(ii) If there is a non-uniform magnetic field, then the radius of the spiral is larger in regions of weak field and smaller in regions of high field.

A velocity selector

A charged particle in crossed electric and magnetic fields can still have constant velocity motion. This occurs if the electric and magnetic forces balance perfectly. This special case can be used as a velocity selector. That is, if we want to select particles with speed v from a set of particles with different speeds, we can do so by arranging crossed electric and magnetic fields in the correct manner. Here is how to do it: Choose $\vec{v}$, $\vec{B}$ and $\vec{E}$ all perpendicular to each other. The direction of the magnetic force is then either parallel or antiparallel with the electric force. If we choose the electric force to be antiparallel to the magnetic force, we can arrange for them to cancel each other. This is achieved if,

$$qE = qvB \tag{12}$$

Therefore particles with speed v will travel without deflection, provided,

$$v = E/B \tag{13}$$

Notice that this does not depend on the charge or the mass of the particles. However the selectivity does depend on the charge and the mass.