PHY481 - Lecture 24

Chapter 7,8 of PS, Chapter 5 of Griffiths

A. Cyclotrons - Another Lorentz force law problem

One of the most notable features of motion in a constant magnetic field is that the classical angular frequency $\omega_c = qB/m$ does not depend on radius of the orbit. Cyclotron's take advantage of this fact to accelerate charged particles using a constant frequency electric field. The radius of the orbit is given by, $R = v/\omega$, so as the particle's velocity is increased, the radius increases, however the frequency does not change. This is not true once the particle becomes relativistic in which case there is a relativistic correction, $\omega = qB/(\gamma m) = \omega_c(1-(v/c)^2)^{1/2}$.

B. Current and current density

The relation between current and current density is the same as the relation between any flux variable and the flux density, so that,

$$\phi_E = \int_S \vec{E}\cdot d\vec{A}; \quad \phi_B = \int_S \vec{B}\cdot d\vec{A}; \quad i = \int_S \vec{j}\cdot d\vec{A} \tag{1}$$

For any "flow" variable, there is an associated flux density variable. The electric field and magnetic field are quite strange as in classical physics nothing is flowing, the fields are static. These fields act like a real classical flow as in current or mass flux etc.

Now we want to relate current to velocity, as when a charge is moving with velocity $\vec{v}$ it clearly produces a current. We can find the relation easily by using the definition of current as charge per unit time. Consider a small element of area dA with its normal along the $z-axis$. Imagine we have a number density of particles n each with charge q and travelling with velocity v_z along the z-axis. The charge per unit time crossing a surface is given by the number of particles per unit time crossing the surface, times their charge, i.e.

$$\vec{j} = q\frac{dN}{dt} = qn\frac{\delta z}{\delta t}\hat{k} = nqv_z\hat{k} \tag{2}$$

This expression can be written in vector form as $\vec{j} = nq\vec{v}$, which relates the motion of particles to the current density and hence to the current. This is important in the theory of electrical resistivity (the Drude model). Note that in the Drude model the velocity $\vec{v}$ is not the full velocity of the charge carriers. Instead it is the drift velocity - we will discuss this further later in the course.

C. The force between current carrying wires is just the Lorentz force law

We have learned that a moving charge experiences a force which is equal to $\vec{F}_B = q\vec{v} \wedge \vec{B}$. Since current flow corresponds to moving charges, a current carrying wire should also experience a force when it is placed in a perpendicular magnetic field. The force on a straight wire carrying current I is easily derived from the magnetic force. The velocity which contributes to current is the drift velocity, v_d. The current density is given by $n_q q v_d$ and the current is $I = jA = An_q q v_d$. The force on each charge is then, on average,

$$\vec{F}_B = q\vec{v}_D \wedge \vec{B} = \frac{\vec{I}}{An_q} \wedge \vec{B} \tag{3}$$

The force on the wire is proportional to the number of charge carriers are travelling through the magnetic field, i.e. $N = n_q V = n_q Al$ where l is the length of the wire. We thus have,

$$\vec{F}_l = N\vec{F}_B = n_q Al \frac{\vec{I}}{An_q} \wedge \vec{B} = l\vec{I} \wedge \vec{B} \tag{4}$$

The direction of the current flow can be kept in either I or in the length l. If we have a small piece of wire in a magnetic field we may then write,

$$d\vec{F} = Id\vec{l} \wedge \vec{B} = IlBsin(\theta) \tag{5}$$

Here we have assigned the direction of current flow to the vector direction of the wire. The direction of the force is given by the right hand rule, as is the case for the magnetic force $\vec{F}_B$. First we show that this is consistent with the Ampere's result for the force between two parallel current carrying wires, $|F| = \mu_0 i_1 i_2/(2\pi d)$.

D. The vector potential

We have learned that several of the tools of magnetostatics,

$$\vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j}; \quad \vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B}) \tag{6}$$

Now we learn another method, the use of the vector potential.

First how do we introduce it. In magnetostatics the magnetic field is divergence free, and we have the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{F}) = 0$ for any vector function $\vec{F}$, therefor if we write $\vec{B} = \vec{\nabla} \wedge \vec{A}$, then we ensure that the magnetic field is divergence free. This is how we can introduce this new function that we call the vector potential, $\vec{A}$. OK but does it help us. To see what we get from this, plug this expression into the differential form of Ampere's law and we find that,

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \mu_0 \vec{j} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} \tag{7}$$

where we used a vector identity to write the last expression on the RHS. This still looks messy, however now note that the choice of $\vec{A}$ is not unique as we can write $\vec{A} + \vec{\nabla} f$ and $\vec{\nabla} \wedge \vec{A}$ is unaltered. We can then choose the scalar function f to help solve problems. In electrostatics a convenient choice is $\vec{\nabla} \cdot \vec{A} = 0$ as this removes the first term in the last expression on the RHS of Eq. (7). In this so-called Coulomb gauge, the equation for the vector potential becomes,

$$\nabla^2 \vec{A} = -\mu_0 \vec{j}; \qquad \text{Coulomb gauge.} \tag{8}$$

This is just Poisson's equation for each of the components of $\vec{A}$. Hey that's terrific we can use what we learned in electrostatics, but for three separate equations. In regions of space where there are no current sources we have three separate Laplace's equations, so the solutions we know for the Cartesion, Spherical Polar, and Cylindrical co-ordinates can be used. We can also write down the superposition result,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J(\vec{r'}) d^3 r'}{|\vec{r} - \vec{r'}|}; \quad \text{while in electrostatics} \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r'}) d^3 r'}{|\vec{r} - \vec{r'}|} \tag{9}$$

The tools that we learned for electrostatics thus carry over to magnetostatics with the relatively minor extension that now we are working with vectors rather than scalars. In addition to find the magnetic field from the vector potential we have to take a curl instead of taking a (-) gradient as occurs in finding the electric field from the electrostatic potential.