

PHY481 - Lecture 25

Chapter 7,8 of PS, Chapter 5 of Griffiths

A. Basic equations of magnetostatics

The analogy with electrostatics is very close, with the major difference that the sources of magnetic fields in magnetostatics are current rings and the intrinsic magnetic moment of elementary particles. $+ \rightarrow N$, $- \rightarrow S$ enables us to draw magnetic field lines in analogy with electrostatics.

Basic equations,

$$\oint \vec{B} \cdot d\vec{a} = 0; \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (\text{no name}) \quad (1)$$

which implies that $\vec{B} = \vec{\nabla} \wedge \vec{A}$. In magnetostatics we choose the Coulomb gauge, $\text{div} \vec{A} = 0$.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i; \quad \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} \quad (\text{Ampere's law}) \quad (2)$$

Boundary conditions on \vec{B} ,

$$B_n^{ext} - B_n^{int} = 0 \quad (\text{from (1)}); \quad B_t^{ext} - B_t^{int} = \mu_0 K \quad (\text{from (2)}) \quad (3)$$

Superposition laws

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{id\vec{l} \wedge \hat{r}}{r^2} \quad \text{or} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}') \wedge (\vec{r} - \vec{r}') d^3r'}{|\vec{r} - \vec{r}'|^3} \quad (4)$$

and

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}') d^3r'}{|\vec{r} - \vec{r}'|} \quad \text{or} \quad \nabla^2 \vec{A} = -\mu_0 \vec{j} \quad (5)$$

This last equation looks like 3 Poisson's equations. In parts of space where there are no current sources, we have 3 Laplace's equations. We can then solve these equations with various boundary conditions. To do this we need the boundary conditions on the vector potential, that are,

$$\vec{A}^{ext} = \vec{A}^{int}; \quad \frac{\partial \vec{A}}{\partial n} \Big|_{ext} - \frac{\partial \vec{A}}{\partial n} \Big|_{ext} = -\mu_0 \vec{K} \quad (6)$$

Another useful relation,

$$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \cdot \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \phi_B \quad (7)$$

In some cases this makes it easy to find the vector potential from the magnetic field, e.g. a solenoid, where $B = \mu_0 ni$, so that $\phi_B = Ba$, and $A(r) = \mu_0 nia / (2\pi r)$. The direction is the

same as the direction of the current (from superposition law above).

B. Magnetic dipoles

There are two sources of magnetic dipoles: currents and intrinsic magnet moments of elementary particles. In either case, we have a magnetic dipole moment \vec{m} that gives the magnitude and direction of the magnetic dipole. This magnetic dipole moment is directly analogous to the electric dipole moment \vec{p} , and we have,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3}; \quad \vec{\tau} = \vec{m} \cdot \vec{B}; \quad U = -\vec{m} \cdot \vec{B} \quad (8)$$

Two key new things here are: (i) that the magnetic moment of a current ring is given by,

$$\vec{m} = i\vec{a}, \quad (9)$$

where \vec{a} is the area of the current loop and the direction is normal to the current loop. (ii)

In addition we have a new relation for the vector potential of a magnetic dipole,

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \wedge \hat{r}}{r^2}. \quad (10)$$

It is a useful exercise to show that $\vec{\nabla} \wedge \vec{A}$ gives the dipole field formula (this was an assigned problem (2.8)).

C. Lorentz force law and force on current carrying wire

A charged particle of charge q with velocity \vec{v} moving in a magnetic field \vec{B} and electric field \vec{E} experiences a force given by,

$$\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B}) \quad (\text{Lorentz force law}) \quad (11)$$

A wire segment carrying current i and with vector length $d\vec{l}$ in a magnetic field \vec{B} experiences a force,

$$d\vec{F} = id\vec{l} \wedge \vec{B} \quad (12)$$

D. Additional equations related to current

The current and current density are related through,

$$i = \int \vec{j} \cdot d\vec{A} \quad (13)$$

The current density is related to the charge density, n , and charge velocity, \vec{v} , through,

$$\vec{j} = nq\vec{v} \quad (14)$$

The continuity equation, which is a statement of charge conservation, is,

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} \quad (15)$$

To prove this write,

$$Q = \int \rho(\vec{r}') d^3r' \quad \text{so that} \quad i_{out} = -\frac{dQ}{dt} = \oint \vec{j} \cdot d\vec{a} \quad (16)$$

Therefore,

$$\int \frac{\partial \rho(\vec{r}')}{\partial t} d^3r' = - \int d^3r' \vec{\nabla} \cdot \vec{j} \quad (17)$$

so that, leading to the continuity equation. The negative sign is due to the fact that if i_{out} is positive, then dQ/dt is negative.

E. A couple of useful math things

In many *EM* calculations, the quantity $1/|\vec{r} - \vec{r}'|$ appears, so it is useful to know about some of its basic mathematical properties, including,

$$\vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (18)$$

that is just the relation between the potential and electric field in electrostatics. Also,

$$\vec{\nabla}^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi\delta(\vec{r} - \vec{r}') \quad (19)$$

This also follows from the potential due to a point charge located at position \vec{r}' . The multipole expansion is,

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} \sum_{l=0} \left(\frac{r'}{r} \right)^l P_l(\cos\theta') \quad (20)$$

where θ' is the angle between \vec{r} and \vec{r}' .

F. DC electric motors

Note that the formula $d\vec{F} = id\vec{l} \wedge d\vec{B}$ explains DC electric motors, such as Faraday's motor. Real DC motors also need a commutator so the current flow direction is maintained. A simple commutator consists of two metal D's with potential difference V between them, however real DC motors have at least 3 sections to reduce dead spots in the drive.