PHY481 - Lecture 26

Sections 8.3, 10.2.1, 11.5 of PS, Section 7.3, 9.2 of Griffiths

A. The effect of time varying electric and magnetic fields

Electrostatics is the study of time independent (static) electric fields. Magnetostatics is the study of time independent (static) magnetic fields. The source of static electric fields is the fundamental charge, q of elementary particles, for example the electron charge e. The sources of magnetic fields are: (i) The fundamental or intrinsic magnetic moment of elementary particles - the magnitude of the magnetic moment of elementary particles, $g\frac{q}{2m}s$, where g is the g-factor, m is the mass and s is the spin of the particle. e.g. for the electron $s = \hbar/2$ and $q = e$. (ii) Steady state currents (DC).

Note that the intrinsic magnetic moment formula $g\frac{q}{2m}s$ applies to elementary particles, ie quarks, leptons, photons, gluons etc. For example the neutron is neutral but it has a magnetic moment because it is a composite particle made up of quarks. In fact the magnetic moment of the neutron is negative so that the neutron spin aligns in the opposite direction to the magnetic field. The fact the neutron has a finite magnetic moment and yet is not charged is very important to its use in scattering studies of materials, particularly polymeric materials, biological materials and magnetic materials.

The effects of time varying electric and magnetic fields are included through two quite simple physical observations:

1. A time varying magnetic flux leads to an induced electric field (Faraday)
2. A time varying electric flux leads to an induced magnetic field (Maxwell's displacement current term)

Faraday's law takes the forms,

$$induced\ emf = \epsilon = -\frac{\partial \phi_B}{\partial t} \quad \text{or} \quad \int \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t} \quad \text{or} \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

This equation states that a time varying magnetic flux induces a voltage or electromotive force. The minus sign on the RHS is understood through Lenz's law which states that the induced emf acts to oppose the change in flux.

Example of Faraday's law

Consider the simplest time-varying magnetic field $\vec{B} = (c_1 + c_2 t)\hat{k}$ that is uniform in space i.e. does not dependent on x, y, z. Place a circular conducting ring in this magnetic field with its unit normal along the z-direction i.e. the vector area of the ring is $\vec{a} = \pi R^2 \hat{k}$. Faraday's law states that there is an induced emf in the ring, given by,

$$emf = -\frac{\partial \phi_B}{\partial t} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} = -\frac{\partial (c_1 + c_2 t) \pi R^2}{\partial t} = -c_2 \pi R^2 \tag{2}$$

Some questions:

- where is this emf?

- does the emf exist if there is no conducting current ring?

- what direction is the emf?

- which direction does the current flow?

Maxwell's displacement current term is an additional source term added to Ampere's law of magnetostatics namely,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t}; \quad \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{(Ampere − Maxwell law)} \tag{3}$$

Maxwell's term describes the fact that a time varying electric field induces a magnetic field. Maxwell noticed that when a capacitor is charging, there is a logical inconsistency in Ampere's law. To understand this inconsistency, consider an initially uncharged capacitor connected to a voltage source at $t = 0$. For simplicity, we consider a parallel plate capacitor. Current begins to flow in the circuit at $t = 0$, charging up the capacitor. Now we can construct a loop around the wire in the circuit. However this loop does not really enclose the current in the wire. The loop can pass through the capacitor without cutting the wire. Therefore when the capacitor is charging, Amperes law would state that there is no enclosed current and hence the magnetic field is zero. This is wrong. There is a magnetic field produced by the current. Maxwell resolved this difficulty by adding a new term which includes the effect of the electric field which builds up between the capacitor plates. His idea was to related this electric field to the current flowing the circuit. From Gauss' law, we have,

$$\frac{d\phi_E}{dt} = \frac{1}{\epsilon_0} \frac{dq}{dt} \tag{4}$$

or the "Maxwell displacement current" is,

$$i_d = \epsilon_0 \frac{d\phi_E}{dt} \tag{5}$$

This relates the current flowing into the capacitor to the electric field between the plates. Maxwell realized that if Amperes law is modified to,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 (i + i_d) = \mu_0 i + \mu_0 \epsilon_0 \frac{d\phi_E}{dt} \tag{6}$$

then the logical inconsistency in the case of a charging capacitor is removed. This extra term is called the displacement current as it has the same dimensions as the true current in the circuit. His insight was brilliant as this equation is correct in general.

Example 30-7

A parallel plate capacitor is being charged at $i = 1C/s$. If the plates are circular with radius, $R = 0.1m$, and are separated by $d = 1cm$, find the magnetic field as a function of distance from the central axis of the capacitor.

Consider a circular loop or radius r centered on the axis of the capacitor and lying parallel to the plates. From Ampere's law we have,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\phi_E}{dt} \tag{7}$$

The electric flux through the loop is given by,

$$\phi_E = \pi r^2 E \tag{8}$$

so the rate of change of the electric flux is,

$$\frac{d\phi_E}{dt} = \pi r^2 \frac{dE}{dt} = \frac{\pi r^2}{d}\frac{dV}{dt} = \frac{\pi r^2}{Cd}\frac{dQ}{dt} = \frac{\pi r^2}{\epsilon_0 A}\frac{dQ}{dt} \tag{9}$$

Here we have used the relation for a capacitor $Q = CV$ and the relation between electric field and voltage for a parallel plate capacitor $E = V/d$, and the expression for the capacitance of a parallel plate capacitor $C = \epsilon_0 A/d$. Evaluating the path integral for the magnetic field and equating it to the displacement current term, we then have,

$$2\pi r B(r) = \mu_0 \epsilon_0 \frac{\pi r^2}{\epsilon_0 A}\frac{dQ}{dt} \tag{10}$$

or

$$B(r) = \frac{r\mu_0}{2A}\frac{dQ}{dt} = \frac{\mu_0 r}{2\pi R^2}\frac{dQ}{dt} = \frac{\mu_0 i r}{2\pi R^2} \quad r < R \tag{11}$$

For $r > R$, the magnetic field is $\frac{\mu_0 i}{2\pi r}$.

B. Electromagnetic waves

Optics, Electricity and Magnetism were considered to be unrelated subjects prior to 1820. In 1820 Ampere and Oersted demonstrated that electric current influences magnets and founded the field of magnetostatics. Faraday extended this to include the effect of time varying magnetic fields. However, it was not until 1864 that optics and other electromagnetic waves were unified and their relation to electricity and magnetism was made clear, by James Clerk Maxwell, though addition of his displacement current term, and then solving the equations to show that they predict wave motion that describes EM waves across all frequencies. This prediction was subsequently confirmed by Hertz.

The demonstration of EM waves is actually quite straightforward. In free space, there are no wires so the term $\mu_0\vec{j}$ is not needed, and there are no charges so we remove the term ρ/ϵ_0 from Gauss's law, so that,

$$\vec{\nabla}\cdot\vec{E}=0;\quad \vec{\nabla}\cdot\vec{B}=0;\quad \vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t};\quad \vec{\nabla}\wedge\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\quad \text{free space} \tag{12}$$

If we take a time derivative of Ampere's law and use Faraday's law, we find,

$$\nabla\wedge(\nabla\wedge\vec{E})=-\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t} \tag{13}$$

If we take a time derivative of Faraday's law then use Ampere's law, we find,

$$\nabla\wedge(\nabla\wedge\vec{B})=-\frac{\partial^2\vec{B}}{\partial t} \tag{14}$$

An identity that is easy to prove (e.g. using Mathematica!) is,

$$\nabla\wedge(\nabla\wedge\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F} \tag{15}$$

Now note that in free space $\nabla\cdot\vec{E}=\nabla\cdot\vec{B}=0$, Using these expressions in Eqs. (13) and (14), we find,

$$\nabla^2\vec{E}=\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \tag{16}$$

$$\nabla^2\vec{B}=\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2} \tag{17}$$

These are both wave equations, which just means that they have solutions which are of the form,

$$E_x(x,y,z)=E_0cos(kz-\omega t+\phi)\quad;\quad E_y=0\quad;\quad E_z=0 \tag{18}$$

The sin function also works, and solutions like this apply to the y direction and to the z direction. We have to choose the solutions to fit the equations and the initial conditions. Eq. (18) describes an EM wave that travels in the $z-direction$ and whose electric field oscillates in the x direction. Similar solutions exist for waves travelling in the $x-direction$

and in the $y - direction$. We have freedom to choose the direction of motion and also the direction of polarization, as well as the phase in each case - there is thus a lot of freedom.

It is essential to first understand one component of this general solution, so lets consider a function E_x, traveling in the z-direction, as given above. We take $E_y = E_z = 0$ so this is a linearly polarized wave that is polarized in the x direction. If we subsitute this expression into Eq.(16), we find,

$$-k^2 = -\mu_0\epsilon_0\omega^2 \tag{19}$$

or, using $k = 2\pi/\lambda$, $\omega = 2\pi f$, $c = f\lambda$,

$$\frac{1}{\mu_0\epsilon_0} = (f\lambda)^2 = c^2 \tag{20}$$

This demonstrates that the velocity of the wave is,

$$c = \frac{1}{(\mu_0\epsilon_0)^{1/2}} = 2.99 \times 10^8 m/s \tag{21}$$

This discovery is considered one of the most important of the 19th century and provides the unification of optics, electricity and magnetism. Since $E_y = E_z = 0$, we have,

$$(\nabla \wedge \vec{B})_x = \mu_0\epsilon_0\frac{\partial E_x}{\partial t} = \omega E_0\mu_0\epsilon_0 sin(kz - \omega t + \phi) \tag{22}$$

We also have,

$$(\nabla \wedge \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \to \frac{-\partial B_y}{\partial z} = \omega E_0\mu_0\epsilon_0 sin(kz - \omega t + \phi) \tag{23}$$

If we integrate this expression with respect to z, we find,

$$B_y = \frac{E_0}{c}cos(kz - \omega t + \phi) = \frac{E_x}{c} \tag{24}$$

This expression shows that the magnetic field is oscillating in phase with the electric field.

The electric field oscillates in the x direction, the magnetic field oscillates in the y-direction and the wave travels in the z-direction. From Maxwell's equations in free space it is evident that $\vec{E}$, $\vec{B}$ and the direction of motion are mutually perpendicular. These are the basic properties of EM waves and the full solution is found by superposition. The direction of motion is usually denoted by $\hat{k}$ and we have $\hat{B} = \hat{k} \wedge \hat{E}$.

The directions we choose above are not special and we can write the cosine and sine parts of the solution in a unified form. We also define $\vec{k}$ as the vector direction of motion (NOT the unit vector in the z-direction, which we will call $\hat{z}$, and write for a monochromatic, linearly polarized EM wave,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}; \quad \vec{B}(\vec{r}, t) = \frac{1}{c}\hat{k} \wedge \vec{E}. \tag{25}$$