

PHY481 - Lecture 27

Sections 10.1-10.3 PS, 7.2 of Griffiths

In the last lecture we went through the most remarkable consequence of including dynamics in the study of electricity and magnetism - EM waves. It is important to emphasize that although the relations between time varying magnetic flux and induced emf (Faraday's law) and between time varying electric flux and induced magnetic field (Maxwell displacement term), both involve loops, these terms exist regardless of whether there is a material loop in the experiment. Today we look at some of the remarkable effects of Faraday's law that occur when there are materials in the experiment.

A. Faraday's law and the Lorentz force law: A moving conducting rod in a constant field

Faraday's law and the Lorentz force law are closely related as can be seen by considering a conducting wire moving through a constant magnetic field.

Consider a uniform and constant magnetic field, B , directed along the z-axis. Now consider moving a conducting rod, of length l , which is directed along the y-direction at constant speed v along the x-direction. First we use Faraday's law to show that a motional emf is developed between the ends of this wire. To find this emf, consider a rectangular loop which is composed of a side of length L lying on the y-axis (centered at the origin), the moving piece of wire, and the two sides which join them to form the rectangle. These two joining sides have length $L = vt$, where we assume that the conducting rod starts at the origin at time $t = 0$. The rate of change of the flux is given by,

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -Bl\frac{dL}{dt} = -Blv \quad (1)$$

This emf is induced around the loop, however the only conducting part of the loop is the piece of wire. The charges in the piece of wire move until the voltage drop between the ends of the wire just balance the induced emf, ie. $V_{wire} = Blv$, so that the voltage inside the rod is zero. If the rod was made of an insulating material, the charges would not move and there would be a voltage across the rod.

The behavior described above can also be understood from the Lorentz force law. Consider the wire to be composed of negatively charge carriers. The motion of these carriers in the magnetic field leads to the Lorentz force law $F_B = -evB$. The charges build up at the

ends of the wire until the induced electric field produces a force on the conductors which just balances the magnetic force. We then have,

$$eE = e \frac{V_{wire}}{l} = evB \quad \text{so that} \quad V_{wire} = Blv \quad (2)$$

as found using Faraday's law. A more general derivation of from the Lorentz force law as is given in PS 10.1.2.

An important deduction from this example is that when an observer moves through a constant magnetic field, the observer sees an electric field. The effect of transformation to a moving co-ordinate system is then very interesting and is believed to be the way in which Einstein first started thinking about relativity.

B. A conducting loop and magnetic drag

Now consider a square loop of wire which lies in the x-y plane, and where each side has length l . Consider that the half space $x < 0$ contains a constant and uniform magnetic, B , directed along the positive z-axis. Now consider the situation in which the square loop is initially within the B field and it is drawn out of the B field at velocity v along the x-axis. If the loop has resistance R find the induced current.

The rate of change of flux is given by,

$$\mathcal{E} = -\frac{d\phi_B}{dt} = Blv \quad (3)$$

The induced emf is then Blv . The current in the loop is thus,

$$i = \frac{Blv}{R} \quad (4)$$

The direction of the current is to oppose the changing flux. Since the flux is decreasing, the current flows counterclockwise in the x-y plane. This produces an induced flux in the z-direction which opposes the changing flux induced by the motion of the loop out of the uniform B field. Note that the induced current is small if the resistance of the loop is large, while the induced current is large if the resistance is small.

In cases where an induced current flows, eg. a conducting loop, there is dissipation in the loop. This energy loss must be equal to the work done by an external force, but what is the origin of the force - it is the Lorentz force and the most convenient form in this problem is

$d\vec{F} = id\vec{l} \wedge \vec{B}$. In the simple example above, the rod direction is perpendicular to the field, so we find that the magnitude of the magnetic drag force is,

$$F_{drag} = ilB = \frac{B^2 l^2 v}{R} \quad \text{direction opposite } \vec{v} \quad (5)$$

In general a moving magnet near a conductor leads to induced “eddy” currents that oppose the motion. This effect can be strong leading to strong braking effects, e.g. when a magnet is dropped down a copper tube. The external force provides power given by,

$$P_{external} = \vec{F}_{external} \cdot \vec{v} = iLB = \frac{B^2 L^2 v^2}{R} = \frac{\mathcal{E}^2}{R} \quad (6)$$

The mechanical power that is supplied is dissipated as resistive losses due to current flow in the loop. Surprisingly, the smaller the resistance the larger the resistive losses and the greater the drag force.

A more complex example

Another popular type of problem is to change the way in which the loop is removed from the field, for example consider the case where the loop above has mass m and falls under gravity out of the field. In that case, taking the x-direction to be the direction downward in the gravitational field,

$$m \frac{dv}{dt} = mg - F_{drag} = mg - \frac{B^2 l^2 v}{R}; \quad \text{or} \quad \frac{dv}{dt} = g - \alpha v; \quad \text{where} \quad \alpha = \frac{B^2 l^2}{mR} \quad (7)$$

The latter equation is solved by rewriting it as,

$$\frac{dv}{g - \alpha v} = dt \quad \text{so that} \quad \frac{-1}{\alpha} \ln(g - \alpha v) = t + C; \quad \text{or} \quad \ln(g - v(t)) = -\alpha t - \alpha C \quad (8)$$

solving this equation for $v(t)$ and using the initial condition $v(0) = 0$ gives,

$$v(t) = \frac{g}{\alpha} (1 - e^{-\alpha t}) \quad (9)$$

The terminal velocity occurs when

$$g = \alpha v_{\infty}, \quad \text{or} \quad v_{\infty} = g/\alpha = \frac{mgR}{B^2 l^2} \quad (10)$$

C. Rotating loops - Electric Generators and motors

We consider for illustration the case of a homopolar generator/motor (1831) where a coil is rotated through a constant magnetic field. This generator/motor has the disadvantage

that the coils rotate, requiring the use of commutators, as in the DC motor. Tesla (1880's) introduced the idea of rotating the magnet inside the coils which removes the need for the commutator. Modern AC motors and generators are almost exclusively induction devices having stationary coils and rotating magnets. The principle of operation of these devices is basically the same physics as magnetic braking or drag as described above. However in the case of generators, the current generated is shunted to carry out work in the electrical grid.

Consider a homopolar coil with N turns and area A rotating about its central axis at constant angular speed ω . The angle between the magnetic moment of the loop and the applied magnetic field is ψ . This angle increases as $\psi(t) = \omega t$ due to the constant angular speed of the coil. The rate of change of the magnetic flux is

$$\frac{d\phi_B}{dt} = \frac{d}{dt}(BA\cos(\psi)) = -BA\omega \sin(\omega t) \quad (11)$$

The induced emf is given by Faraday's which gives,

$$\mathcal{E} = -N \frac{d\phi_B}{dt} = NBA\omega \sin(\omega t) \quad (12)$$

The potential energy is given by,

$$U = -\vec{\mu} \cdot \vec{B} = NIAB \cos(\psi) \quad (13)$$

The torque is given by,

$$\tau = \vec{\mu} \wedge \vec{B} = NIAB \sin(\psi) \quad (14)$$

The mechanical power that must be supplied to rotate the coil is,

$$P_{mech} = \vec{\tau} \cdot \vec{\omega} = NIAB\omega \sin(\psi) \quad (15)$$

This must be equal to the electrical power delivered to the load, this is given by,

$$P_{elec} = \frac{\mathcal{E}^2}{R} = \frac{(NBA\omega)^2}{R} \sin^2(\omega t) \quad (16)$$

The average power dissipated is

$$P_{av} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(NBA\omega)^2}{R} \sin^2(\theta) d\theta \quad (17)$$

Using the result that,

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\theta) d\theta = \frac{1}{2} \quad (18)$$

we have,

$$P_{av} = \frac{1}{2} \frac{(NBA\omega)^2}{R} \quad (19)$$

which is the average power delivered by our generator. This is the number you see quoted on your lightbulb.